



# All Formulae & Concept

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# Relations and Functions



## Relations

If set A has **m elements** and set b has **n elements**, then **total no of relation** from set A To Set B is  
 **$= 2^{m \times n}$**

Total no of relation on set A =  **$2^{m \times m}$**

Total no of nonempty relations on Set A =  **$2^{m \times m} - 1$**



# Relations and Functions

## Empty Relation Or Void Relation

A relation  $R$  in a set  $A$  is called an empty relation, if no element of  $A$  is related to any element of  $A$  and we denote such a relation by  $\phi$ .

**Example :** Let  $A = \{1, 2, 3, 4\}$  and let  $R$  be a relation in  $A$ , given by  $R = \{(a, b) : a + b = 20\}$ .

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## Universal Relation

A relation  $R$  in a set  $A$  is called an universal relation, if each element of  $A$  is related to every element of  $A$ .

**Example :** Let  $A = \{1, 2, 3, 4\}$  and let  $R$  be a relation in  $A$ , given by  $R = \{(a, b) : a + b > 0\}$ .



# Relations and Functions

## Identity Relation

A relation  $R$  in a set  $A$  is called an identity relation, where  $R = \{(a, a), a \in A\}$ .

**Example :** Let  $A = \{1, 2, 3, 4\}$  and let  $R$  be a relation in  $A$ , given by  $R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$ .

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# Relations and Functions



## Reflexive Relation

A relation  $R$  in a set  $A$  is called a Reflexive relation, if  $(a, a) \in R$ , for all  $a \in A$ .

Total no of reflexive relation is  $2^{n(n-1)}$

### Example :

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Let  $A = \{1, 2, 3, 4\}$  and let  $R$  be a relation in  $A$ , given by

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2)\}$$

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (1, 3), (3, 1)\}$$

# Relations and Functions



## Symmetric Relation

A relation  $R$  in a set  $A$  is called a symmetric relation, if  $(a, b) \in R$ , then  $(b, a) \in R$  for all  $a, b \in A$ .

Total no of symmetric relation is  $2^{\frac{n(n+1)}{2}}$

## Example :

Let  $A = \{1, 2, 3, 4\}$  and let  $R$  be a relation in  $A$ , given by ATDB.uno

$$R = \{(1, 1), (2, 2), (3, 3)\}$$

$$R = \{(1, 2), (2, 1), (3, 3)\}$$

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (1, 3), (3, 1), (3, 2)\}.$$

# Relations and Functions



## Transitive Relation

A relation  $R$  in a set  $A$  is called a transitive relation, if  $(a, b) \in R$  and  $(b, c) \in R$ , then  $(a, c) \in R$  for all  $a, b, c \in A$

Or

$(a, b) \in R$  and  $(b, c) \notin R$  for all  $a, b, c \in A$  ATDB.uno

## Example :

Let  $\{1, 2, 3, 4\}$  and let  $R$  be a relation in  $A$ , given by

$R = \{(1, 1), (2, 2), (3, 3)\}$ . (According to second condition)

$R = \{(1, 2), (2, 1), (1, 1), (2, 2)\}$ . (According to first condition)

$R = \{(2, 3), (3, 1), (1, 3), (3, 1), (3, 2), (3, 3), (2, 2), (1, 1)\}$ .



# Relations and Functions

## Equivalence Relation

A relation  $R$  in a set  $A$  is said to be an equivalence relation, if it is reflexive, symmetric and transitive.

## Functions :

A relation  $f$  from a set  $A$  to a set  $B$  is said to be a function, if every element of set  $A$  has one and only one image in set  $B$ .

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**Note :** No two distinct ordered pairs in  $f$  have the same first element.

Number of functions from  $A$  to  $B = n^m$

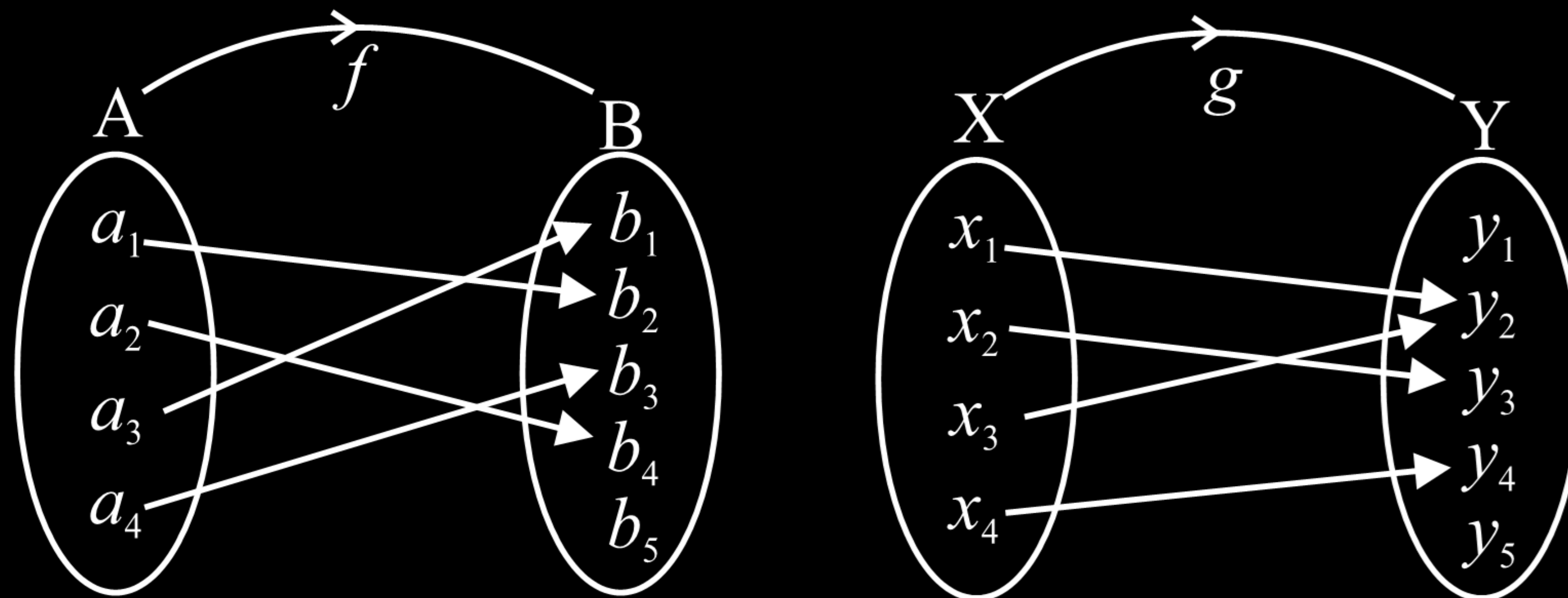
# Relations and Functions



## One-one function (injection) :

A function  $f: A \rightarrow B$  is said to be a one-one function or an injection, if different elements of A have different images in B.

Number of one-one functions from A to B =  $\begin{cases} {}^n P_m, & \text{if } m \leq n \\ 0, & \text{if } m > n \end{cases}$



# Relations and Functions



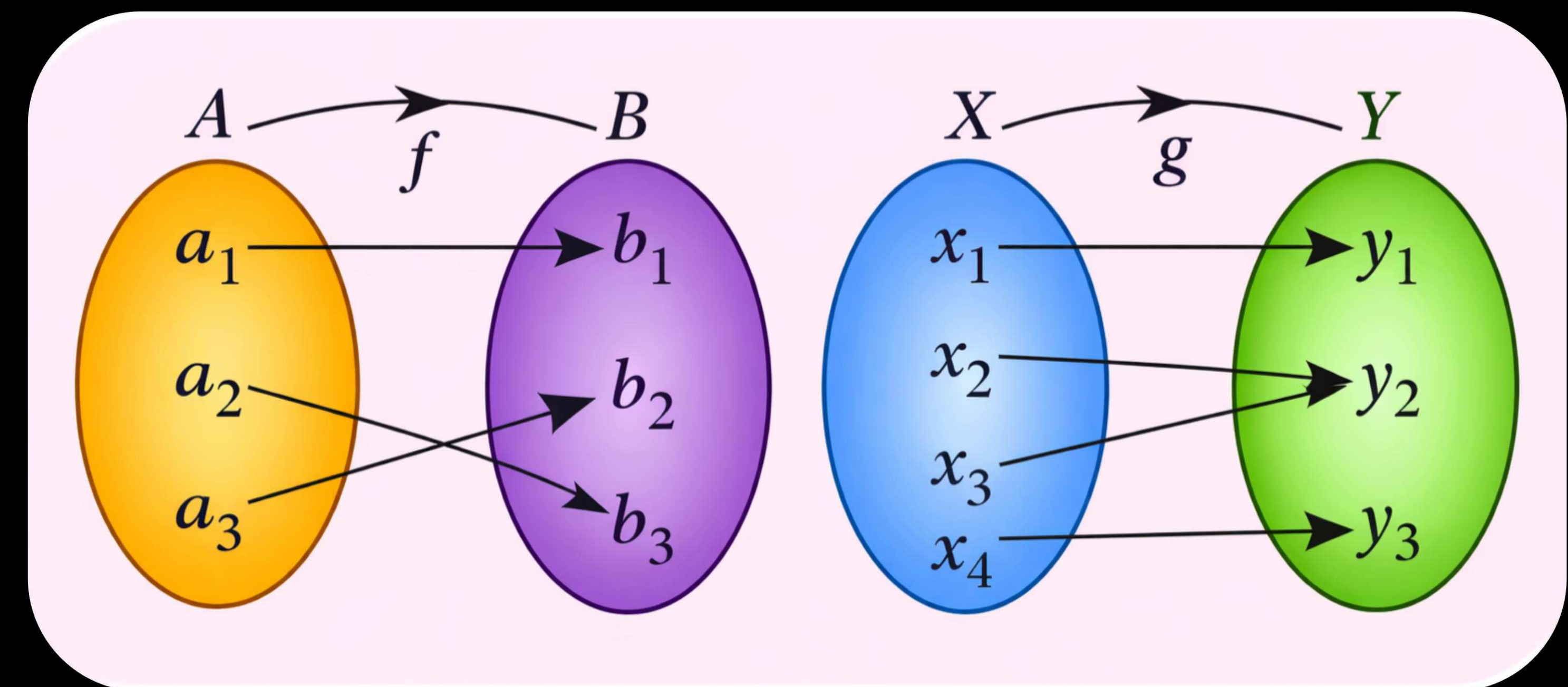
## Onto function (surjection) :

A function  $f: A \rightarrow B$  is onto if each element of B has its preimage in A.

In other words, **Range of  $f$  = Co-domain of  $f$ .**

Number of onto functions from A to B =  $\sum_{r=1}^n (-1)^{n-r} {}^n C_r r^m$

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# Relations and Functions



## Definition :

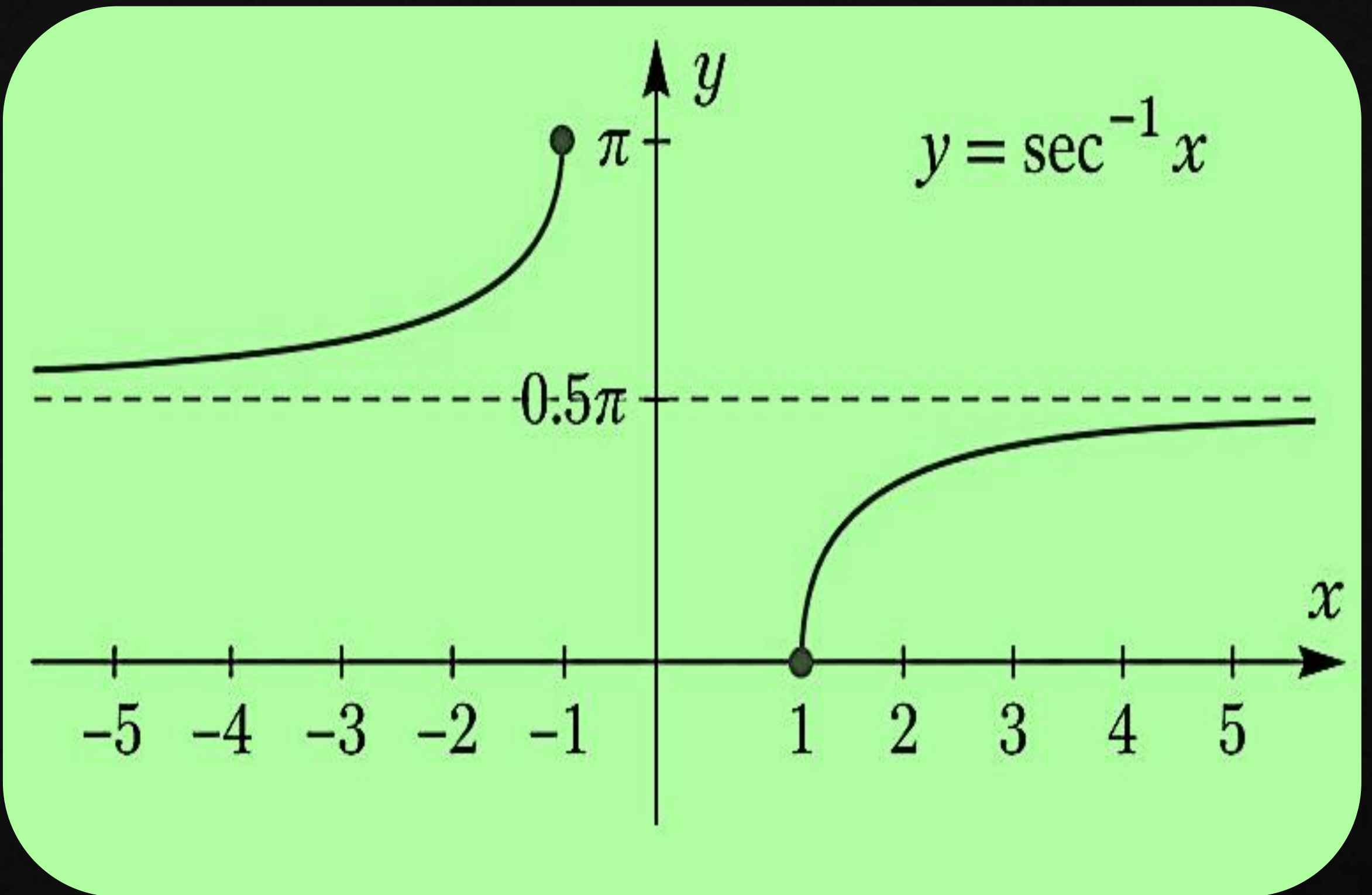
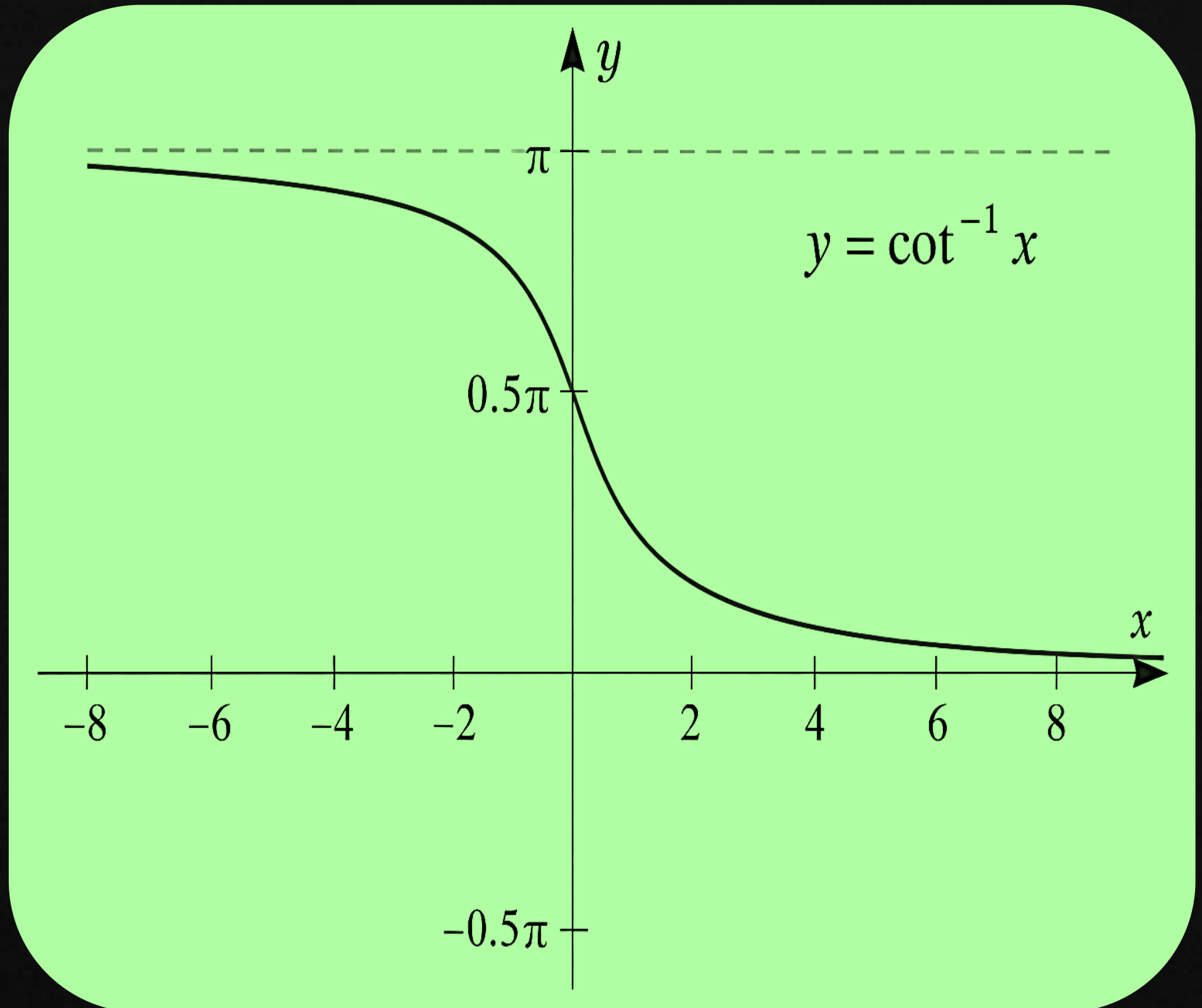
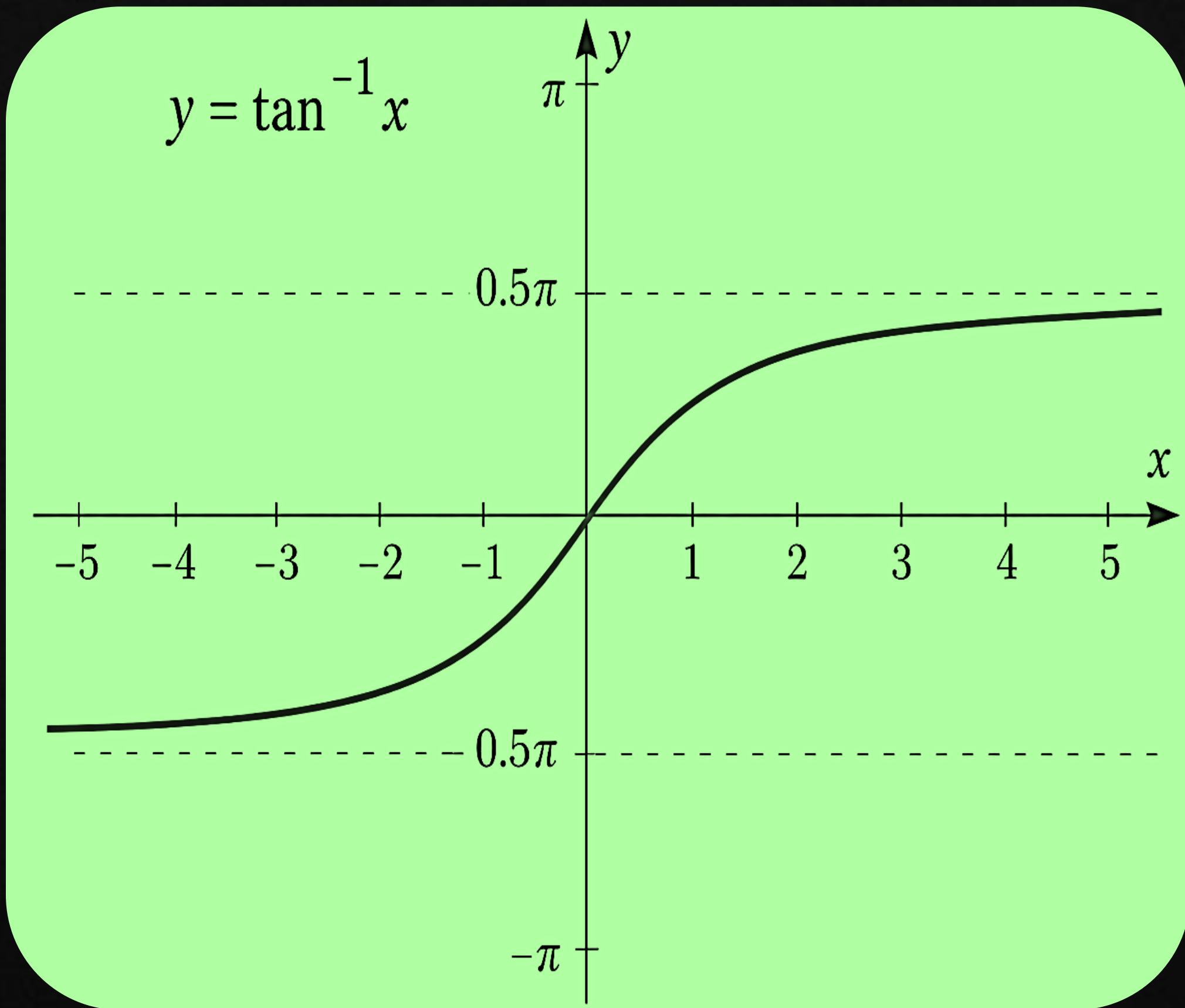
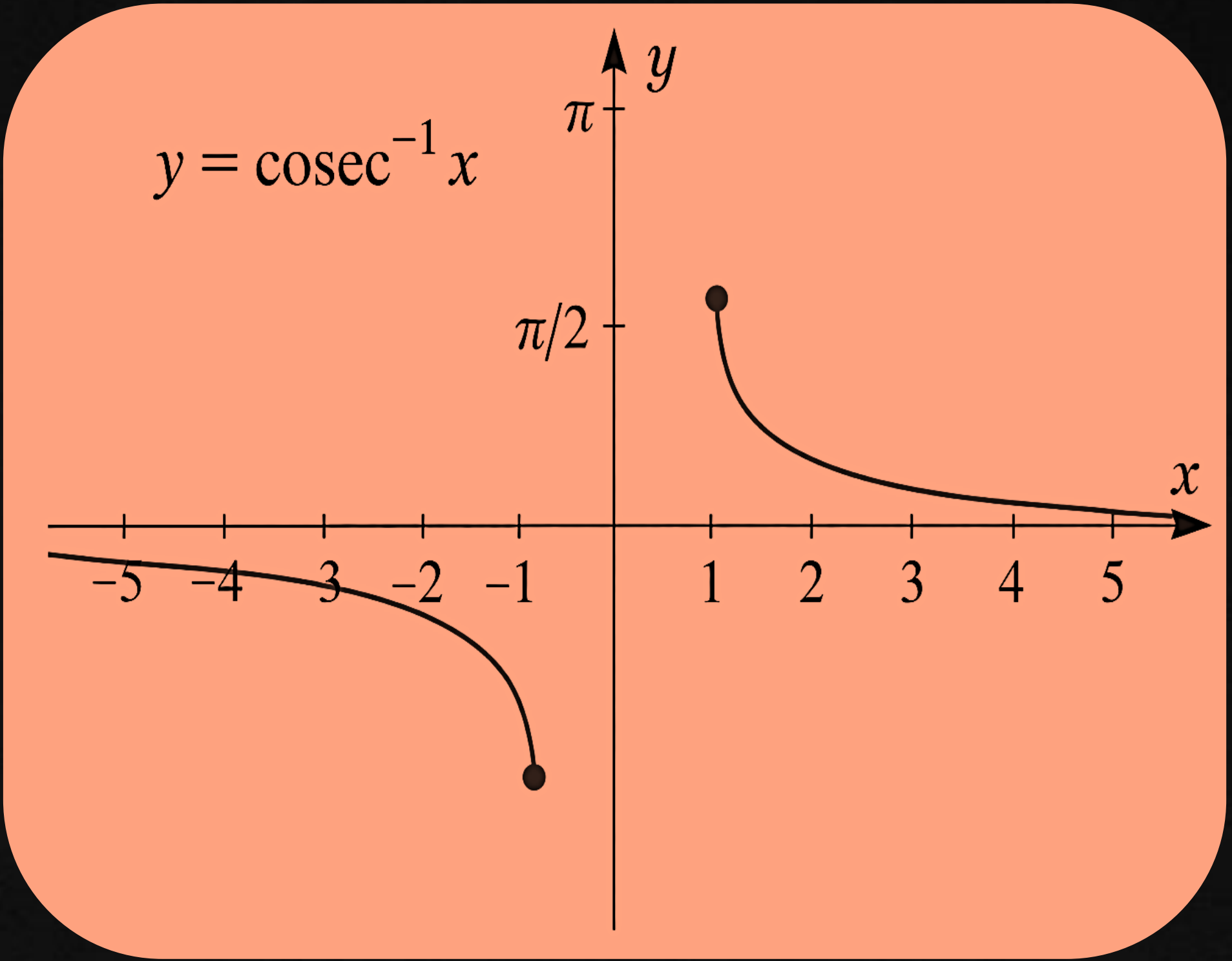
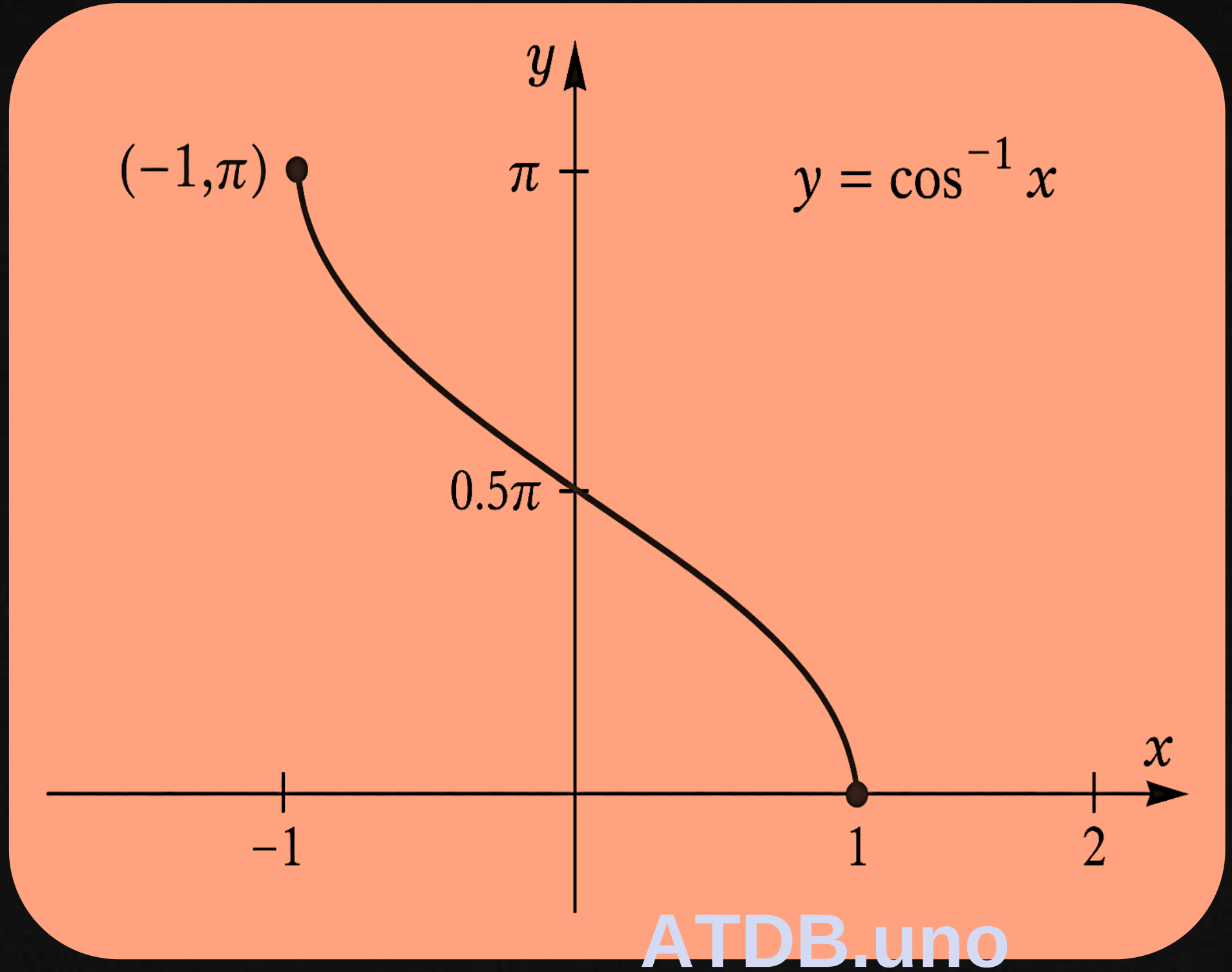
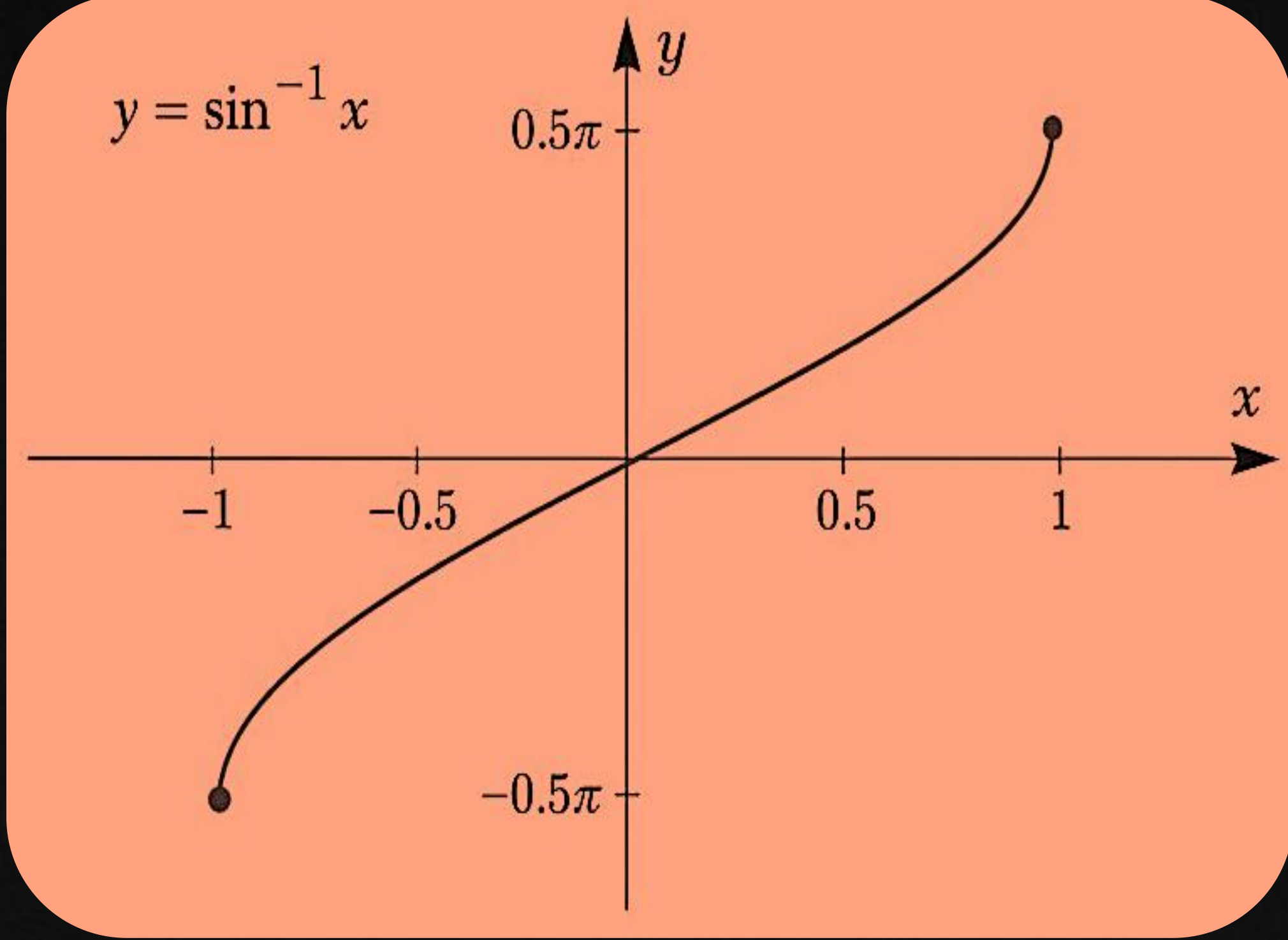
A function  $f : X \rightarrow Y$  is said to be one-one and onto (or bijective), if  $f$  is both one-one and onto.

Total no of bijective functions from  $A$  to  $B = \begin{cases} m!, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases}$

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ITF



ITF



| Function                         | Domain        | Range                                                  |
|----------------------------------|---------------|--------------------------------------------------------|
| $y = \sin^{-1}x$                 | $[-1, 1]$     | $\left[ \frac{-\pi}{2}, \frac{\pi}{2} \right]$         |
| $y = \cos^{-1}x$                 | $[-1, 1]$     | $[ 0, \pi ]$                                           |
| $y = \tan^{-1}x$                 | R             | $\left( \frac{-\pi}{2}, \frac{\pi}{2} \right)$         |
| $y = \cot^{-1}x$                 | R             | $(0, \pi)$                                             |
| $y = \sec^{-1}x$                 | $R - (-1, 1)$ | $[0, \pi] - \left\{ \frac{\pi}{2} \right\}$            |
| $y = \operatorname{cosec}^{-1}x$ | $R - (-1,1)$  | $\left[ \frac{-\pi}{2}, \frac{\pi}{2} \right] - \{0\}$ |

# ITF



- when  $x \in [-1, 1]$   $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ .
- when  $x \in \mathbb{R}$   $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$ .
- when  $x \in \mathbb{R} - (-1, 1)$   $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$ .
- $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left( \frac{x+y}{1-xy} \right)$ , if  $xy < 1$  **ATDB.uno**
- $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \left( \frac{x-y}{1+xy} \right)$ , if  $xy > -1$
- $2 \tan^{-1} x = \tan^{-1} \left( \frac{2x}{1-x^2} \right)$

- $\sin^{-1}(\sin x) = x$ , when  $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\cos^{-1}(\cos x) = x$ , when  $x \in [0, \pi]$
- $\tan^{-1}(\tan x) = x$ , when  $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- $\cot^{-1}(\cot x) = x$ , when  $x \in (0, \pi)$
- $\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x$ , when  $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
- $\sec^{-1}(\sec x) = x$ , when  $x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$
- $\sin(\sin^{-1} x) = x$ , when  $x \in [-1, 1]$
- $\cos(\cos^{-1} x) = x$ , when  $x \in [-1, 1]$
- $\tan(\tan^{-1} x) = x$ , when  $x \in \mathbb{R}$
- $\cot(\cot^{-1} x) = x$ , when  $x \in \mathbb{R}$
- $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$ , when  $x \in \mathbb{R} - (-1, 1)$
- $\sec(\sec^{-1} x) = x$ , when  $x \in \mathbb{R} - (-1, 1)$
- $\sin^{-1}(-x) = -\sin^{-1} x$ , when  $x \in [-1, 1]$
- $\cos^{-1}(-x) = \pi - \cos^{-1} x$ , when  $x \in [-1, 1]$
- $\tan^{-1}(-x) = -\tan^{-1} x$ , when  $x \in \mathbb{R}$
- $\cot^{-1}(-x) = \pi - \cot^{-1} x$ , when  $x \in \mathbb{R}$
- $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$ , when  $x \in \mathbb{R} - (-1, 1)$
- $\sec^{-1}(-x) = \pi - \sec^{-1} x$ , when  $x \in \mathbb{R} - (-1, 1)$



# Formulae

(i)  $\sin(x + y) = \sin x \cos y + \cos x \sin y$

(ii)  $\sin(x - y) = \sin x \cos y - \cos x \sin y$

(iii)  $\cos(x + y) = \cos x \cos y - \sin x \sin y$

(iv)  $\cos(x - y) = \cos x \cos y + \sin x \sin y$

(v)  $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \cdot \tan y}$

(vi)  $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \cdot \tan y}$

(vii)  $\cot(x + y) = \frac{\cot x \cdot \cot y - 1}{\cot y + \cot x}$

(viii)  $\cot(x - y) = \frac{\cot x \cdot \cot y + 1}{\cot y - \cot x}$

(ix)  $\sin 2x = 2 \sin x \cos x = \frac{2 \tan x}{1 + \tan^2 x}$

(ix)  $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$

(x)  $\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x -$

$1 = 1 - 2 \sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$

(xii)  $\sin 3x = 3 \sin x - 4 \sin^3 x$

(xiii)  $\cos 3x = 4 \cos^3 x - 3 \cos x$

(xiv)  $\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$

(xv)  $\cos x + \cos y = 2 \cos \frac{x + y}{2} \cos \frac{x - y}{2}$

(xvi)  $\cos x - \cos y = 2 \sin \frac{x + y}{2} \sin \frac{y - x}{2}$

(xvii)  $\sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}$

(xviii)  $\sin x - \sin y = 2 \cos \frac{x + y}{2} \sin \frac{x - y}{2}$

(xix)  $2 \sin x \cos y = \sin(x + y) + \sin(x - y)$

(xx)  $2 \cos x \sin y = \sin(x + y) - \sin(x - y)$

(xxi)  $2 \cos x \cos y = \cos(x + y) + \cos(x - y)$

(xxii)  $2 \sin x \sin y = \cos(x - y) - \cos(x + y)$

(xxiii)  $\sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}}$

(xxiv)  $\cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}}$

(xxv)  $\tan \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}}$

sign '+' or '-' will be decided according to the quadrant in which angle lies.

☐  $\cos (-x) = \cos x$

☐  $\sec (-x) = \sec x$

☐  $\sin (-x) = -\sin x$

☐  $\operatorname{Cosec} (-x) = -\operatorname{cosec} x$

☐  $\tan (-x) = -\tan x$

☐  $\cot (-x) = -\cot x$

# Matrices

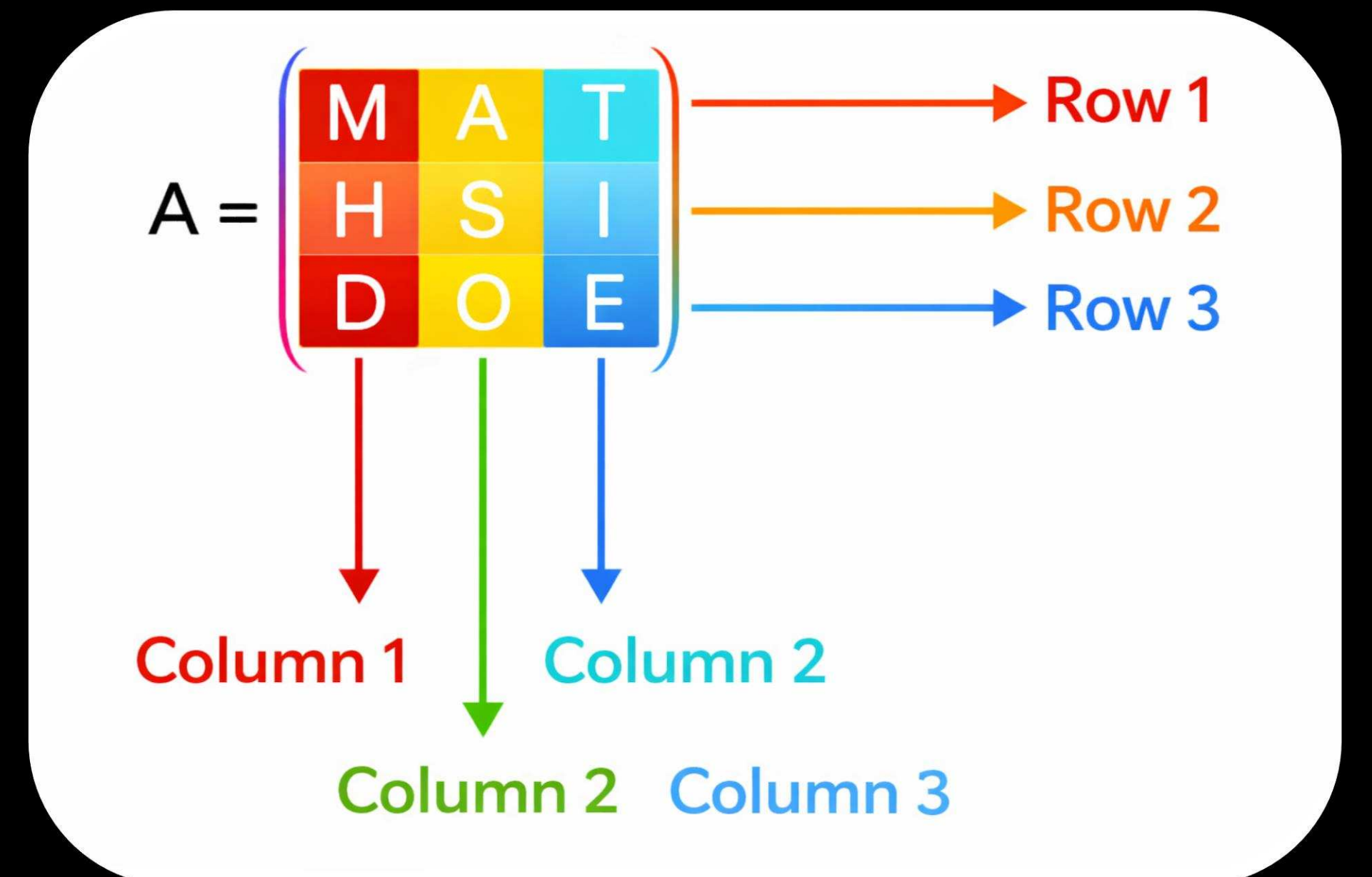


Matrices are defined as a **rectangular arrangement** of numbers or functions. Since it is a rectangular arrangement, it is 2-dimensional.

A two-dimensional matrix consists of the number of rows ( $m$ ) and a number of columns ( $n$ ).

Horizontal ones are **called Rows** and Vertical ones are **called columns**.

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# Matrices

## Order of Matrix :

If a matrix has  $m$  rows and  $n$  columns, then it is known as the matrix of order  $m \times n$ .

Representation of matrix

A general matrix of order  $m \times n$  can be written as

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$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} = [a_{ij}]_{m \times n}, i = 1, 2, \dots, m \quad j = 1, 2, \dots, n$$



# Matrices

## Row Matrix

A matrix having only one row.

**Example :**  $[3 \ 5 \ 7]$

## Column Matrix

A matrix having only one column.

**Example :**  $\begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$

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## Zero / Null Matrix

All elements are zero.

**Example :**  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

## Square Matrix

No. of rows = No. of columns.

**Example :**  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

# Matrices



## Rectangular Matrix

Rows  $\neq$  Columns.

**Example :**  $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$

(  $2 \times 3$  rectangle)

## Diagonal Matrix

Square matrix where non-diagonal elements = 0, diagonal may be non-zero.

ATDB.uno **Example :**  $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

# Matrices



## Scalar Matrix

A diagonal matrix where all diagonal elements are equal.

**Example :**  $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

## Identity (Unit) Matrix

A square matrix where diagonal = 1 and rest = 0.

**Example :**  $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

## Equal Matrices

Two matrices are equal, if

- ✓ Same order
- ✓ Corresponding elements equal

# Matrices



- **TRANPOSE OF A MATRIX :** If  $A = [a_{ij}]_{m \times n}$  be an  $m \times n$  matrix, then the matrix obtained by interchanging the rows and columns of  $A$  is called the transpose of  $A$ .  
Transpose of the matrix  $A$  is denoted by  $A'$  or  $A^T$ .

$$\text{If } A = [a_{ij}]_{m \times n}, \text{ then } A' = [a_{ji}]_{n \times m}$$

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- **SYMMETRIC MATRIX :** A square matrix If  $A = [a_{ij}]$  is said to be symmetric if  $A^T = A$

$$\text{Example : } A = \begin{bmatrix} 2 & 5 & 12 \\ 5 & 7 & 3 \\ 12 & 3 & 6 \end{bmatrix}.$$



# Matrices

- **SKEW-SYMMETRIC MATRIX :** A square matrix  $A = [a_{ij}]$  is said to be skew symmetric matrix if  $A^T = -A$ .

Example :  $A = \begin{bmatrix} 0 & 5 & -12 \\ -5 & 0 & -3 \\ 12 & 3 & 0 \end{bmatrix}$

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# Matrices



- **INVERTIBLE MATRICES :** If  $A$  is a square matrix of order  $m$ , and if there exists another square matrix  $B$  of the same order  $m$ , such that  $AB = BA = I$ , then  $B$  is called the inverse matrix of  $A$  and it is denoted by  $A^{-1}$ . In that case  $A$  is said to be invertible.

## PROPERTIES OF MATRICES :

- $A + B = B + A$ ,  $A - B \neq B - A$ ,  $AB \neq BA$ ,  $(AB)C = A(BC)$
- $(A')' = A$ ,  $AI = IA = A$ ,  $AB = BA = I$ , then  $A^{-1} = B$  and  $B^{-1} = A$
- $AB = 0 \Rightarrow$  it is not necessary that one the matrix is zero
- $A(B + C) = AB + AC$
- $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$  where  $\frac{1}{2}(A + A')$  is symmetric matrix and  $\frac{1}{2}(A - A')$  is skew-symmetric matrices.

# Determinants



To every square matrix we can assign a number **called determinant**

**Determinant :** Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then  $\det(A) = |A| = ad - bc$

Let  $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$ , then  $|A| = a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

For easier calculations, we shall expand the determinant along that row or column which contains maximum number of zeros

# Determinants



□ The area of a triangle whose vertices are  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$ , is  $\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ .

Since area is a positive quantity, we always take the absolute value of the determinant

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□ The area of the triangle formed by three collinear points is zero.

□ Equation of line joining the points  $(x_1, y_1)$  and  $(x_2, y_2)$  is  $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

# Determinants



**Minors :** Minor of an element  $a_{ij}$  of a determinant is the determinant obtained by deleting its  $i$  row and  $j^{\text{th}}$  column in which element  $a_{ij}$  lies. Minor of an element  $a_{ij}$  is denoted by  $M_{ij}$ .

**Co-Factors :** Cofactor of an element  $a_{ij}$ , denoted by  $A_{ij}$  is defined by  $A_{ij} = (-1)^{i+j} \cdot M_{ij}$ , where  $M_{ij}$  is minor of  $a_{ij}$

The value of a determinant  $\Delta =$  sum of the product of elements of any row (or column) with their corresponding cofactors.

If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero. For

**Example :**  $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$

# Determinants



**Adjoint of a Matrix :** The adjoint of a square matrix  $A = [a_{ij}]$  is defined as the transpose of the matrix  $[A_{ij}]$ , where  $A_{ij}$  is the cofactor of the element  $a_{ij}$ . Adjoint of the matrix  $A$  is denoted by  $\text{adj } A$ .

To find adjoint of a  $2 \times 2$  matrix interchange the diagonal elements and change the sign of non diagonal elements.

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# Determinants



**Inverse of a Matrix :** Let A be a square matrix.

$$A^{-1} = \frac{1}{|A|} \text{adj}A$$

**Solution of system of linear equations by using matrix method :**

Let the system of linear equations be  $a_1x + b_1y + c_1z = d_1$ ,  $a_2x + b_2y + c_2z = d_2$   $a_3x + b_3y + c_3z = d_3$

These equations can be written as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$AX = B \Rightarrow X = A^{-1}B$$

# Determinants



- $A^{-1}$  exists, if  $|A| \neq 0$  i.e the solution exists and it is unique.
- The system of equations is said to be consistent if the solution exists.
- if  $|A| = 0$ , then we calculate  $(\text{adj } A)B$ .
- If  $|A| = 0$  and  $(\text{adj } A)B \neq 0$ , (0 being zero matrix), then solution does not exist and the system of equations is called inconsistent. **ATDB.uno**
- If  $|A| = 0$  and  $(\text{adj } A)B = 0$ , then system may be either consistent or inconsistent according as the system have either infinitely many solutions or no solution.

# Determinants



- The matrix  $A$  is singular if  $|A| = 0$
- A square matrix  $A$  is said to be non-singular if  $|A| \neq 0$
- If  $A$  and  $B$  are non-singular matrices of the same order, then  $AB$  and  $BA$  are also non-singular matrices of the same order
- If  $A$  is an invertible matrix, then  $|A| \neq 0$  and  $(A^{-1})^T = (A^T)^{-1}$
- $|\lambda A| = \lambda^n |A|$ , where  $n$  = order of matrix  $A$
- $A(\text{adj}A) = (\text{adj}A)A = |A|I$
- $|\text{adj}A| = |A|^{n-1}$ , where  $n$  = order of matrix  $A$
- $|A(\text{adj}A)| = |A|^n$ , where  $n$  = order of matrix  $A$
- $|AB| = |A||B|$   $(AB)^{-1} = B^{-1}A^{-1}$   $|A^{-1}| = |A|^{-1}$   $|A^T| = |A|$
- If  $A$  and  $B$  are square matrices of the same order, then  $\text{adj}(AB) = \text{adj}B \cdot \text{adj}A$



# Continuity & Differentiability

- A function  $f(x)$  is said to be continuous at  $x = c$  iff  $\lim_{x \rightarrow c} f(x) = f(c)$  i.e.  $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = f(c)$
- $f(x)$  is continuous in  $(a, b)$  iff it is continuous at  $x = c \forall c \in (a, b)$ .
- Modulus functions is Continuous on  $\mathbb{R}$
- Trigonometric functions are continuous in their respective domains.
- Exponential function is continuous on  $\mathbb{R}$  ATDB.uno
- Every polynomial function is continuous on  $\mathbb{R}$ .
- Greatest integer function is continuous on all non-integral real numbers



# Continuity & Differentiability

- Sum, difference, product and quotient of continuous functions are continuous. i.e,  $f$  and  $g$  are continuous functions, then
- $(f \pm g)(x) = f(x) \pm g(x)$  is continuous.  
 $(f \cdot g)(x) = f(x) \cdot g(x)$  is continuous.
- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$  where  $g(x) \neq 0$  is continuous.
- Every differentiable function is continuous. But the converse is not true.

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# Continuity & Differentiability

## Geometrical meaning of continuity:

- (i) Function  $f$  will be continuous at  $x = c$  if there is no break in the graph of the function at the point  $(c, f(c))$ .
- (ii) In an interval, function is said to be continuous if there is no break in the graph of the function in the entire interval.

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## Discontinuity:

The function  $f$  will be discontinuous at  $x = a$  in any of the following cases:

- (i)  $\lim_{x \rightarrow a^-} f(x)$  and  $\lim_{x \rightarrow a^+} f(x)$  exist but are not equal.
- (ii)  $\lim_{x \rightarrow a^-} f(x)$  and  $\lim_{x \rightarrow a^+} f(x)$  exist but are equal but not equal to  $f(a)$ .
- (iii)  $f(a)$  is not defined.



# Continuity & Differentiability

A function  $f(x)$  is derivable or differentiable in its domain iff

$$\lim_{h \rightarrow 0} \frac{f(X - h) - f(x)}{-h} = \lim_{h \rightarrow 0} \frac{f(X + h) - f(x)}{h}, \text{ and is finite.}$$

The value of above limit is denoted by  $f'(x)$  and is called the derivative of  $f(x)$ .

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# Continuity & Differentiability



- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}a(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
- $\frac{d}{dx}(\log x) = \frac{1}{x}$
- $\frac{d}{dx}(a) = 0, a = \text{constant}$
- $\frac{d}{dx}(a^x) = a^x \cdot \log a$

|                                   |                                                                               |
|-----------------------------------|-------------------------------------------------------------------------------|
| $\frac{d}{dx}(\sin x) = \cos x$   | $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$                            |
| $\frac{d}{dx}(\cos x) = -\sin x$  | $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$                                  |
| $\frac{d}{dx}(\tan x) = \sec^2 x$ | $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$ |



# Continuity & Differentiability

## Algebra of derivatives

If  $u, v$  are functions of  $x$ , then

$$(i) \quad \frac{d(u \pm v)}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$$

$$(ii) \quad \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$(iii) \quad \frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

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# Continuity & Differentiability

1.  $\log_b(xy) = \log_b x + \log_b y$

2.  $\log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y$

3.  $\log_b x^n = n \log_b x$

4.  $\log_b x = \frac{\log_c x}{\log_c b}$ , where  $c > 1$

5.  $\log_b x = \frac{1}{\log_x b}$

6.  $\log_b b = 1$  and  $\log_b 1 = 0$

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# Continuity & Differentiability

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

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$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$



# Continuity & Differentiability

- Logarithmic differentiation is a powerful technique to differentiate functions of the form  $f(x) = (u(x))^{v(x)}$ , where both  $f$  and  $u$  need to be positive functions for this technique to make sense.

- Differentiable of a function with respect to another function

Let  $u = f(x)$  and  $v = g(x)$  be two functions of  $x$ , then to find derivative of  $f(x)$  w.r.t  $g(x)$ , i.e.,

find  $\frac{du}{dv}$ , we use the formula  $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$ .

- Second order derivative

$$\frac{d}{dx} \frac{dy}{dx} = \frac{d^2 y}{dx^2} \quad y \quad x \quad y'' \quad y_2 \quad y = f(x)$$

# Application of Derivatives



1. Rate of change in quantities: the rate of change is the ratio of change in a quantity with comparison to change in another quantity.

To find the rate of change in a quantity  $P$  with respect to change in another quantity  $Q$  is  $\frac{dP}{dQ}$

If rate of change in  $P$  with respect to  $Q$  is positive, the quantity  $P$  increases as  $Q$  increase OR  $P$  decreases as  $Q$  decrease. ( $P$  is directly proportional to  $Q$ )

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If rate of change in  $P$  with respect to  $Q$  is negative, the quantity  $P$  decreases as  $Q$  increase OR  $P$  increases as  $Q$  decrease ( $P$  is inversely proportional to  $Q$ )

- If  $x$  denotes the number of items produced, and  $P$  is Profit function,  $\frac{dP}{dx}$  is known as marginal profit.
- If  $x$  denotes the number of items produced, and  $C$  is Cost function,  $\frac{dC}{dx}$  is known as marginal cost.
- If  $x$  denotes the number of items produced, and  $R$  is Revenue function,  $\frac{dR}{dx}$  is known as marginal Revenue.

# Application of Derivatives



- **Increasing decreasing functions :** A function is increasing at a point  $x = a$ , if  $f'(a) \geq 0$

A function is decreasing at a point  $x = a$ , if  $f'(a) \leq 0$

A function is strictly increasing at a point  $x = a$ , if  $f'(a) > 0$

A function is strictly decreasing at a point  $x = a$ , if  $f'(a) < 0$

WORKING RULE to find the interval on which a function is increasing / decreasing

- Write the given function
- Find its derivative
- Find the values of  $x$  (interval) for which the derivative is positive / negative

# Application of Derivatives



## 3. Maxima and Minima:

- (a) The maximum/Minimum value of a function over a closed interval is called Absolute maximum/minimum of the function.
- (b) The maximum/minimum value of a function in the neighborhood of a point, is called local maximum/minimum of the function.
- (c) A point in the domain of a function is called a turning point/critical point, if the derivative of the function becomes zero at that point.
- (d) At each stationery point (other than point of inflection) the function attains a maximum or minimum value.
- (e) The value attains at stationery point may be maximum or minimum. It can be decided by the help of first derivative of the function (FIRST DERIVATIVE TEST) or Second derivative of the function (SECOND DERIVATIVE TEST).



# Application of Derivatives

- (f) **First derivative test :** Function attains maximum value at a stationery point if the value of first derivative is negative at any point to the right of the stationery point.
- (g) **Second derivative test :** Function attains maximum value at a stationery point if the value of second derivative at the stationery point is negative.

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# Application of Derivatives



## **WORKING RULE to find MAXIMUM/MINIMUM value of a function.**

- (i) Identify the function, which is to be maximized or minimized, notice the variables in it.
- (ii) Identify the relation between these variables which is given in the question as a hint.
- (iii) Using above relation, write one variable in terms of other, and express the function which is to be maximized/minimized as a single variable function.  
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- (iv) Find the first and second derivatives of this function.
- (v) Find the values of  $x$  for which the first derivative becomes zero, these points are called critical or turning or stationery points.
- (vi) At each stationery point check the sign of second derivative, if it is positive-Minimum exists at that point, if it is negative-Maximum exists at that point and if it is zero-neither minimum nor maximum exist at that point

# Integrals



|       | Derivatives                                                                                                         | Integrals (Anti Derivatives)                                                            |
|-------|---------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------|
| (i)   | $\frac{d}{dx} \left( \frac{x^{n+1}}{n+1} \right) = x^n ;$<br>Particularly, we note that<br>$\frac{d}{dx} (x) = 1 ;$ | $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$<br><br>$\int dx = x + C$<br>ATDB.uno |
| (ii)  | $\frac{d}{dx} (\sin x) = \cos x ;$                                                                                  | $\int \cos x \, dx = \sin x + C$                                                        |
| (iii) | $\frac{d}{dx} (-\cos x) = \sin x ;$                                                                                 | $\int \sin x \, dx = -\cos x + C$                                                       |
| (iv)  | $\frac{d}{dx} (\tan x) = \sec^2 x ;$                                                                                | $\int \sec^2 x \, dx = \tan x + C$                                                      |

# Integrals



|        | Derivatives                                                               | Integrals (Anti Derivatives)                                                         |
|--------|---------------------------------------------------------------------------|--------------------------------------------------------------------------------------|
| (v)    | $\frac{d}{dx}(-\cot x) = \operatorname{cosec}^2 x ;$                      | $\int \operatorname{cosec}^2 x \, dx = -\cot x + C$                                  |
| (vi)   | $\frac{d}{dx}(\sec x) = \sec x \tan x ;$                                  | $\int \sec x \tan x \, dx = \sec x + C$                                              |
| (vii)  | $\frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cot x ;$ | ATDB.uno<br>$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$ |
| (viii) | $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} ;$                    | $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$                                     |
| (ix)   | $\frac{d}{dx}(-\cos^{-1} x) = \frac{1}{\sqrt{1-x^2}} ;$                   | $\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$                                    |

# Integrals



|        | Derivatives                                           | Integrals (Anti Derivatives)              |
|--------|-------------------------------------------------------|-------------------------------------------|
| (x)    | $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} ;$       | $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$ |
| (xi)   | $\frac{d}{dx}(e^x) = e^x ;$                           | $\int e^x dx = e^x + C$<br>ATDB.uno       |
| (xii)  | $\frac{d}{dx}(\log  x ) = \frac{1}{x} ;$              | $\int \frac{1}{x} dx = \log  x  + C$      |
| (xiii) | $\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x ;$ | $\int a^x dx = \frac{a^x}{\log a} + C$    |



# Integrals

## 3. Methods of integration

- (a) Integration by substitution
- (b) Integration by trigonometric formula
- (c) Method of partial fraction.

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# Integrals

$$(i) \quad \int \tan x \, dx = \log |\sec x| + C$$

$$(ii) \quad \int \cot x \, dx = \log |\sin x| + C$$

$$(iii) \quad \int \sec x \, dx = \log |\sec x + \tan x| + C$$

$$(iv) \quad \int \operatorname{cosec} x \, dx = \log |\operatorname{cosec} x - \cot x| + C \quad \text{ATDB.uno}$$

$$(1) \quad \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

$$(2) \quad \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C$$

$$(3) \quad \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$(4) \quad \int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$(5) \quad \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$$

$$(6) \quad \int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

# Integrals



| S. No. | Form of the rational function                          | Form of the partial fraction                              |
|--------|--------------------------------------------------------|-----------------------------------------------------------|
| 1.     | $\frac{px + q}{(x - a)(x - b)}, a \neq b$              | $\frac{A}{x - a} + \frac{B}{x - b}$                       |
| 2.     | $\frac{px + q}{(x - a)^2}$                             | $\frac{A}{x - a} + \frac{B}{(x - a)^2}$                   |
| 3.     | $\frac{px^2 + qx + r}{(x - a)(x - b)(x - c)}$ ATDB.uno | $\frac{A}{x - a} + \frac{B}{x - b} + \frac{C}{x - c}$     |
| 4.     | $\frac{px^2 + qx + r}{(x - a)^2(x - b)}$               | $\frac{A}{x - a} + \frac{B}{(x - a)^2} + \frac{C}{x - b}$ |
| 5.     | $\frac{px^2 + qx + r}{(x - a)(x^2 + bx + c)}$          | $\frac{A}{x - a} + \frac{Bx + C}{x^2 + bx + c}$           |
|        | where $x^2 + bx + c$ cannot be factorized further      |                                                           |

# Integrals



## (d) Integration by parts

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int [f'(x)\int g(x)dx]dx$$

If we take  $f$  as the first function, easily differentiable and  $g$  as the second function, easily integrable then this formula may be stated as follows:

**Note :** For choosing the first function  $f(x)$  use the abbreviate as ILATE, ILATE- inverse, log, arithmetic, trigonometric function, exponential function.

Integral of the type  $\int e^x[f(x) + f'(x)]dx = e^x f(x) + C$



# Integrals

$$(i) \quad \int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log|x + \sqrt{x^2 - a^2}| + C$$

$$(ii) \quad \int \sqrt{x^2 + a^2} dx = \frac{1}{2} x \sqrt{x^2 + a^2} + \frac{a^2}{2} \log|x + \sqrt{x^2 + a^2}| + C$$

$$(iii) \quad \int \sqrt{a^2 - x^2} dx = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$$

# Integrals



## Some Properties of Definite Integrals

$$\mathbf{P}_0 : \int_a^b f(x)dx = \int_a^b f(t)dt$$

$$\mathbf{P}_1 : \int_a^b f(x)dx = -\int_b^a f(x)dx. \quad \int_a^a f(x)dx = 0$$

$$\mathbf{P}_2 : \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

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# Integrals



$$\mathbf{P}_3 : \int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

$$\mathbf{P}_4 : \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

(Note that  $\mathbf{P}_4$  is a particular case of  $\mathbf{P}_3$ )

$$\mathbf{P}_5 : \int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

$$\mathbf{P}_6 : \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(2a-x) = f(x) \text{ and } 0 \int_0^{2a} f(x)dx = -\int_0^a f(x)dx$$

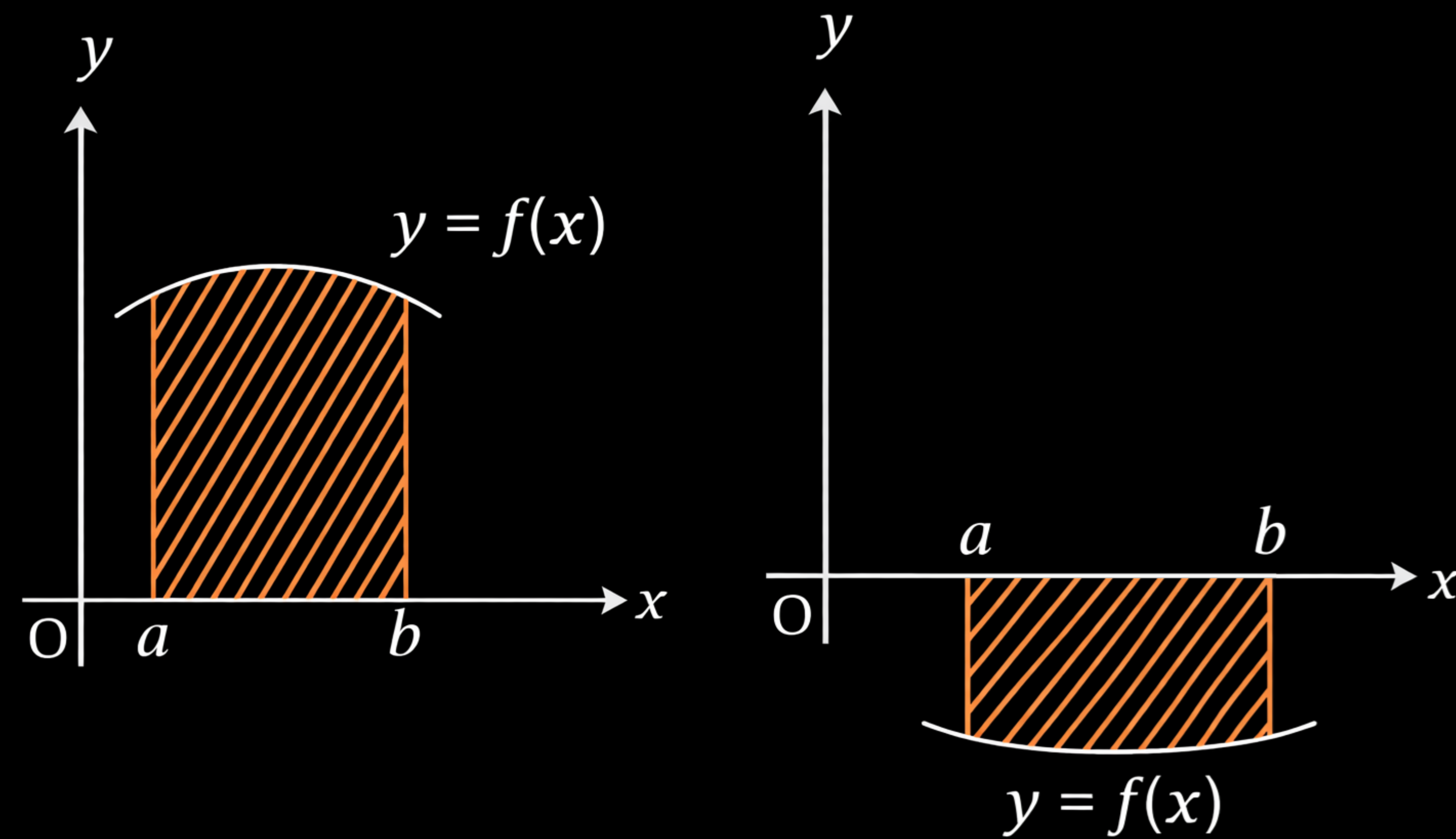
$$\mathbf{P}_7 : \text{(i) } \int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx, \text{ if } f \text{ is an even function, i.e., if } f(-x) = f(x).$$

$$\text{(ii) } \int_{-a}^a f(x)dx = 0, \text{ if } f \text{ is an odd function, i.e., if } f(-x) = -f(x).$$



# Application of Integrals

- Area bounded by the curve  $y = f(x)$ , the  $x$  axis and between the ordinates,  $x = a$  and  $x = b$  is given by



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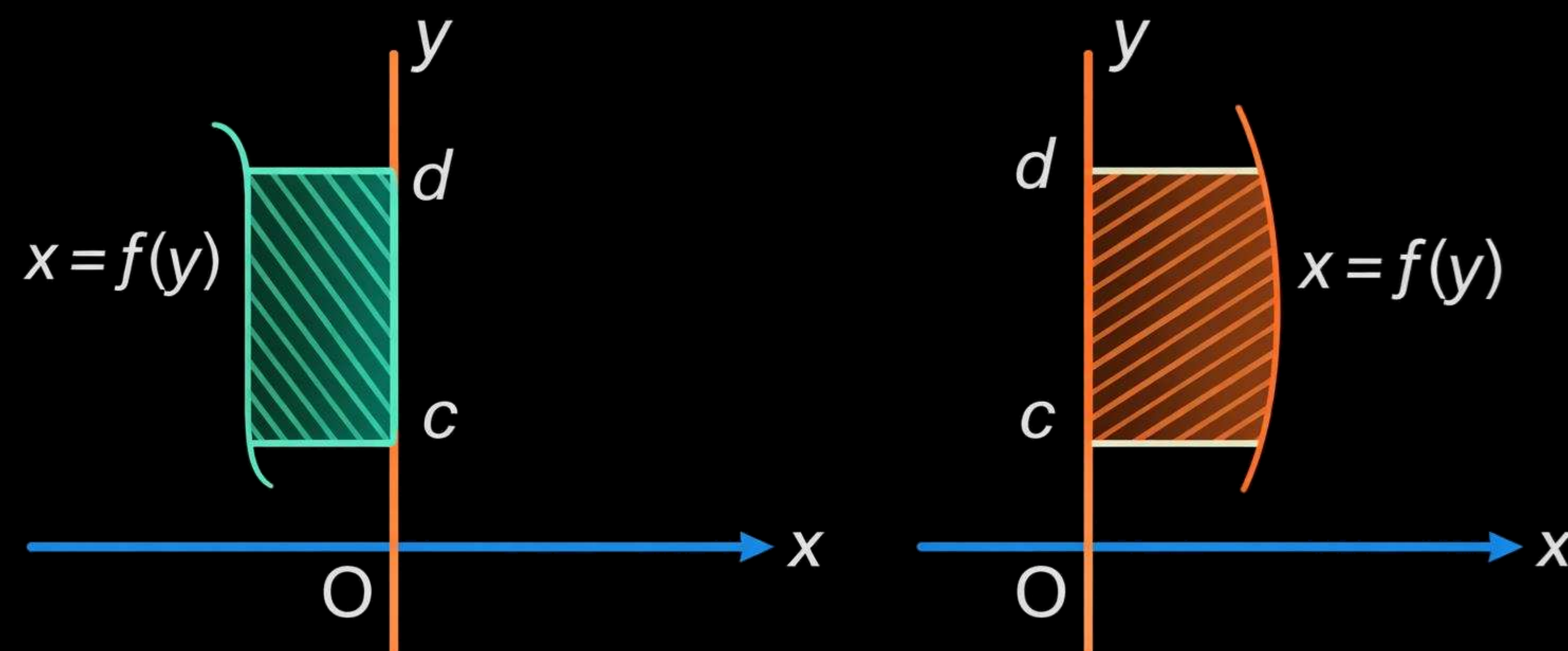


# Application of Integrals

- Area bounded by the curve  $x = f(y)$ , the  $y$ -axis and between the abscissas,  $y = c$  and  $y = d$  is

given by  $\left| \int_c^d f(y) dy \right|$

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# Application of Integrals

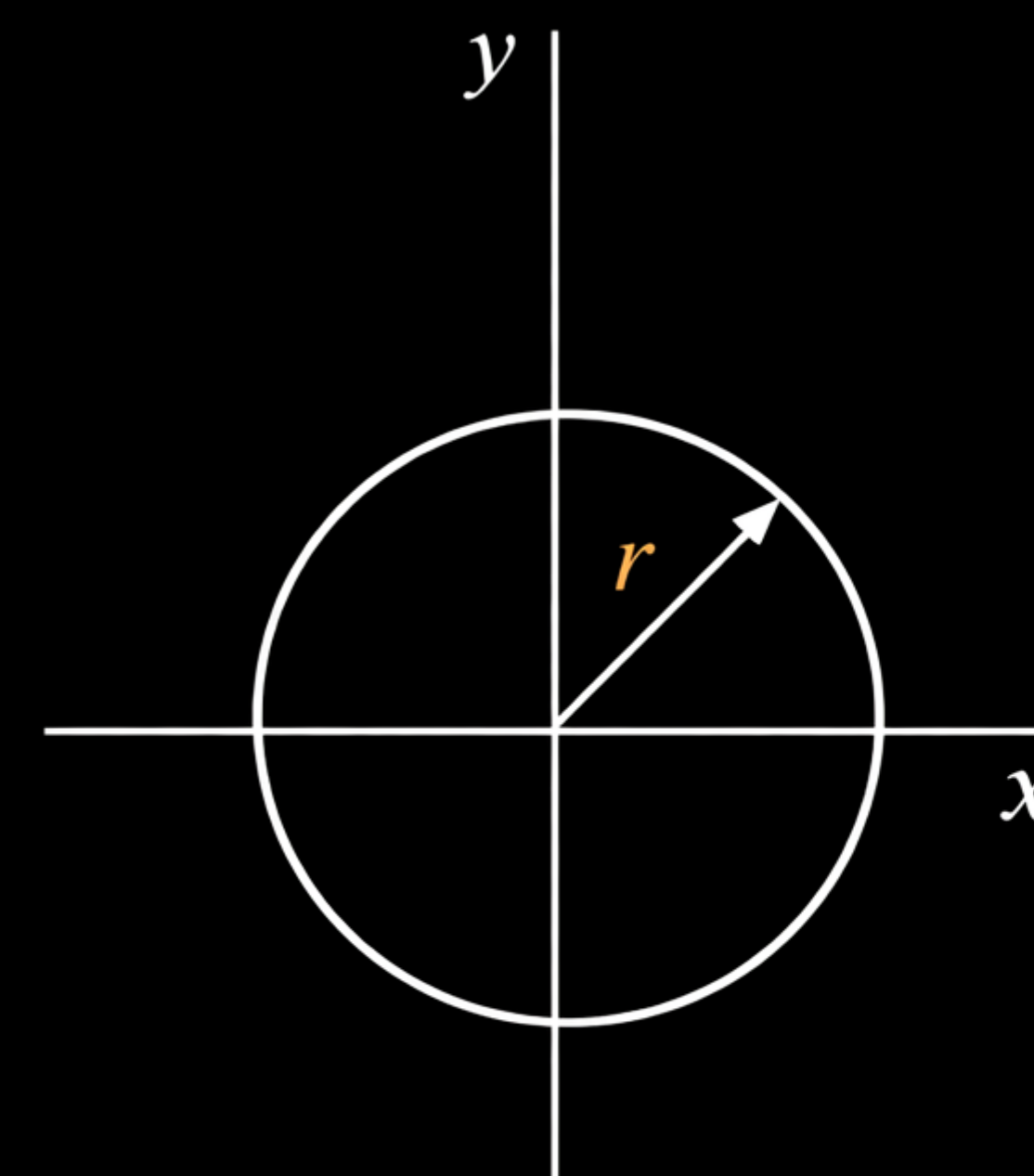


- Integrals are useful in finding areas under simple curves such as lines, circles, parabolas and ellipses.
- Standard Form of Some Important Curves:

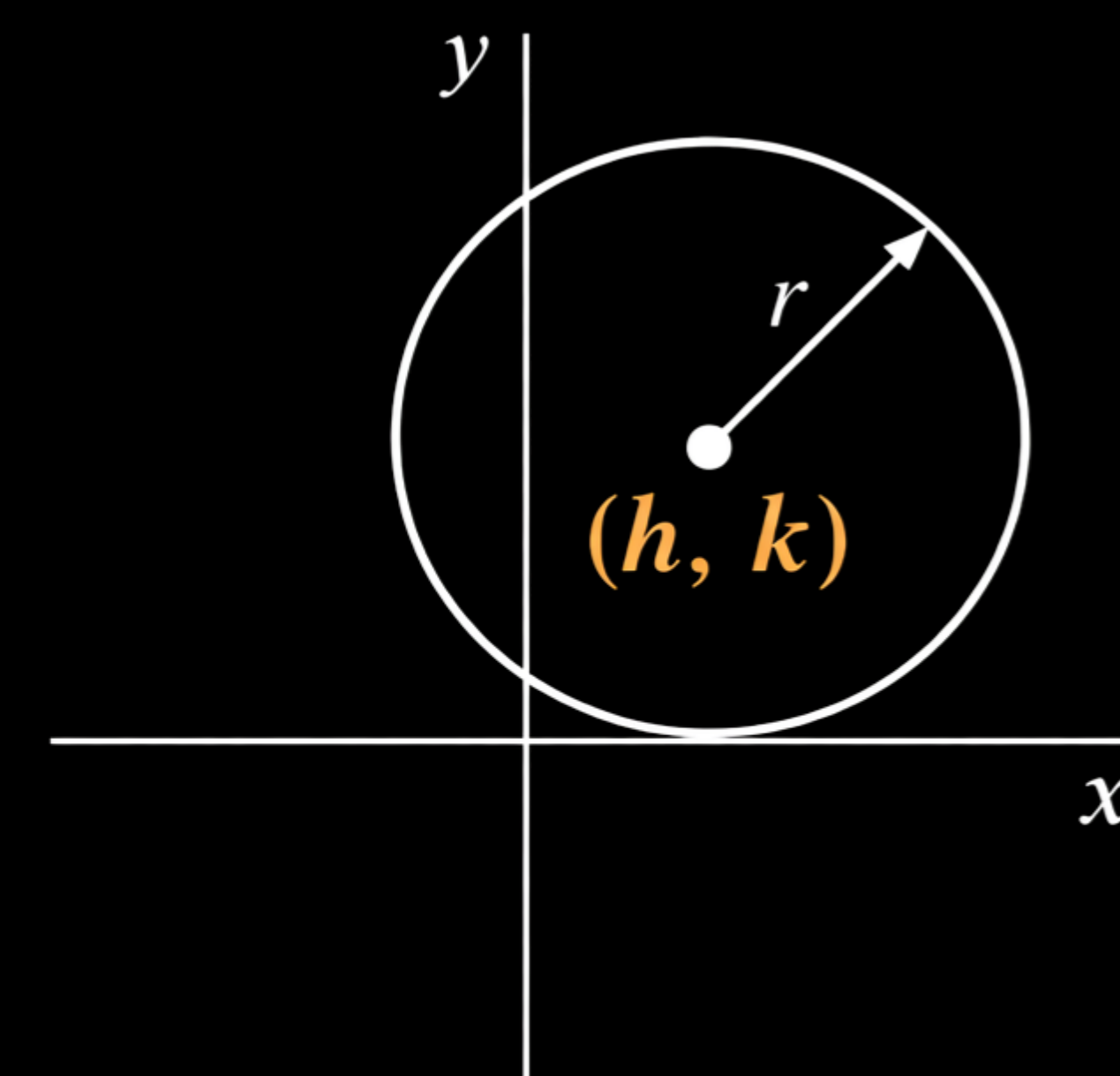
Straight line:  $ax + by = c$

Circle :  $x^2 + y^2 = r^2$  (A circle with centre at origin and radius  $r$ )

$$(x - h)^2 + (y - k)^2 = r^2 \text{ ((h, k) r)}$$



$$x^2 + y^2 = r^2$$

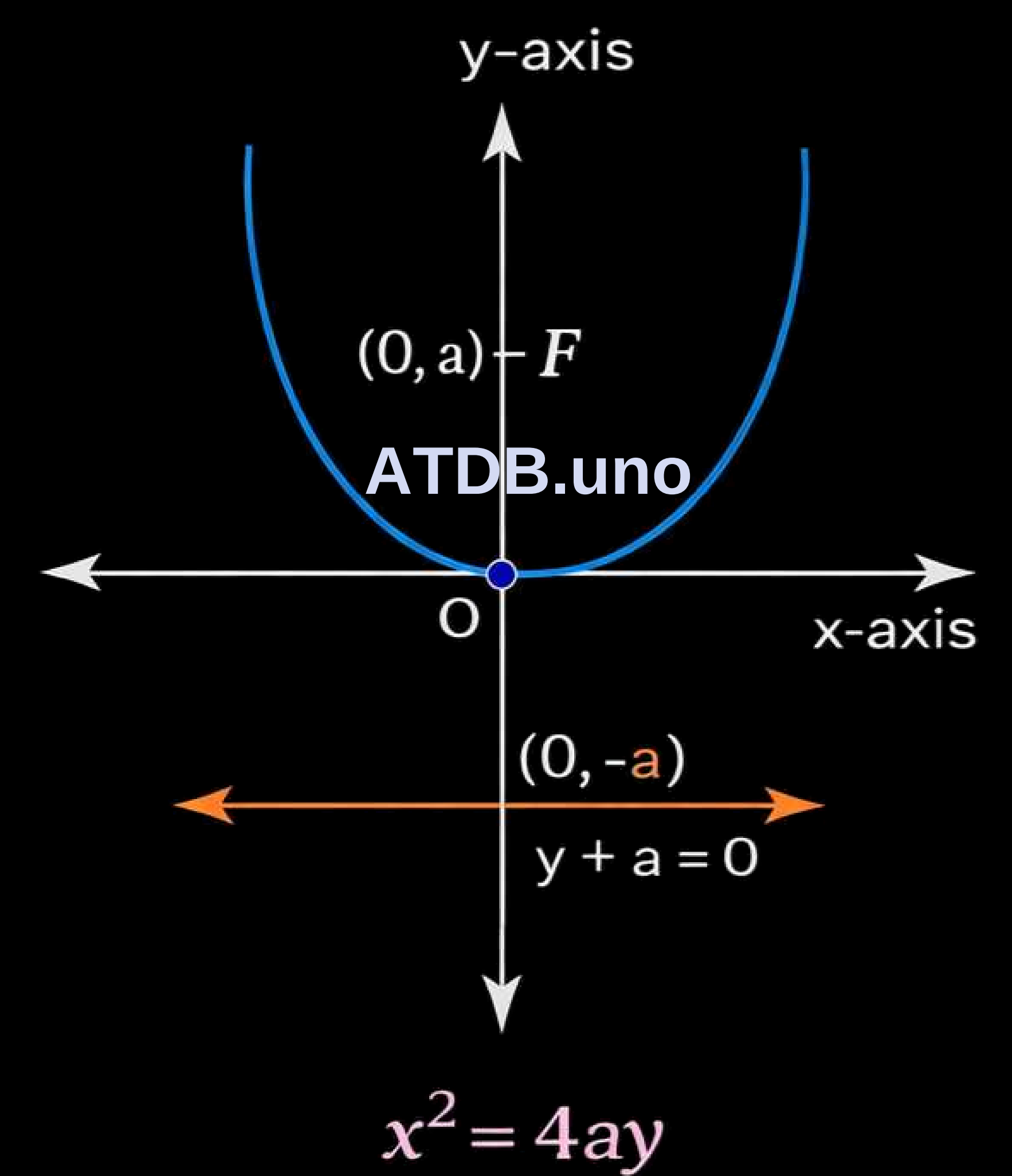
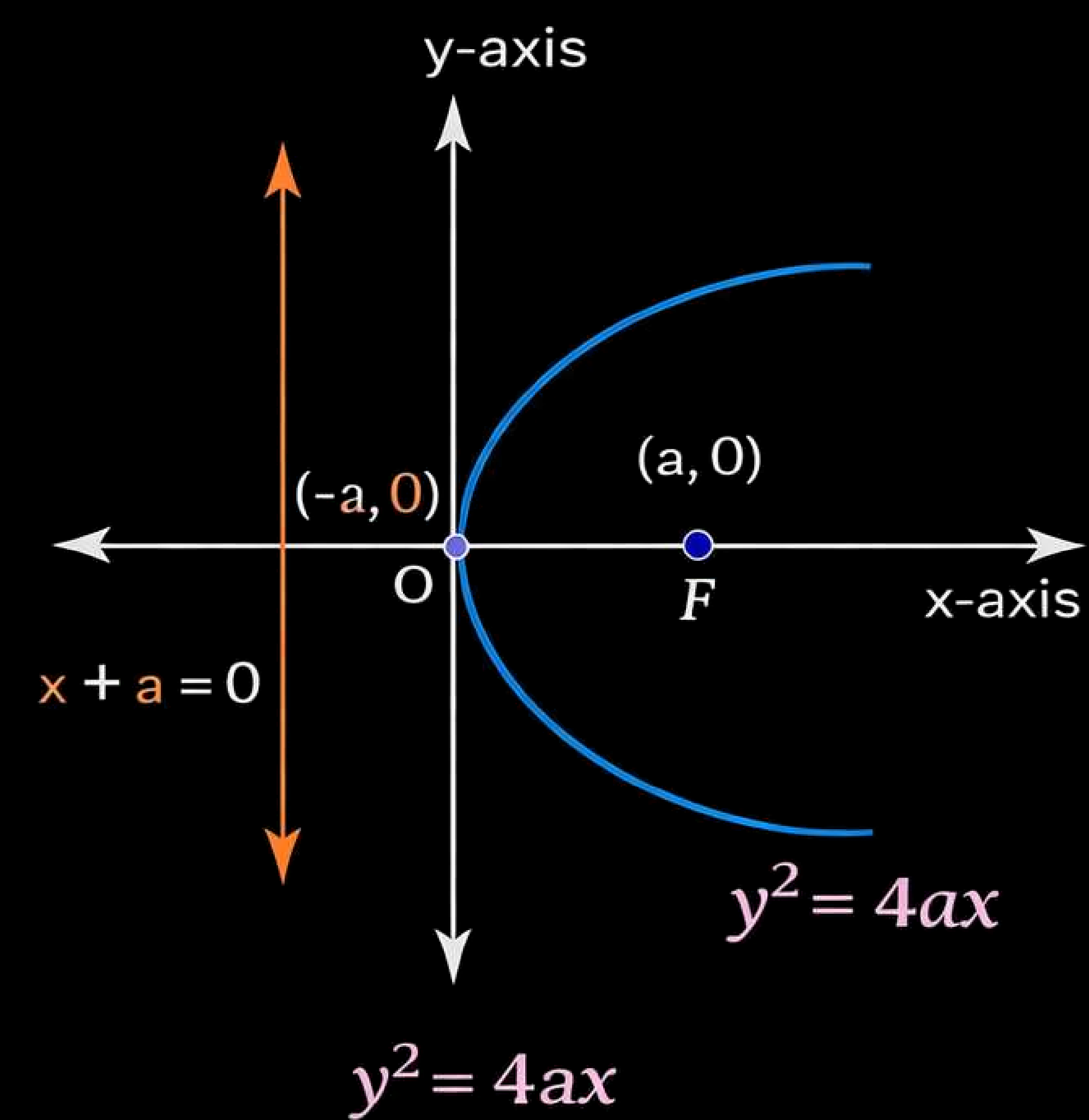


$$(x - h)^2 + (y - k)^2 = r^2$$



# Application of Integrals

## Parabola:





# Application of Integrals

- Area of the region bounded by the curve  $y = f(x)$ ,  $x$ -axis and the lines  $x = a$  and  $x = b$  (where  $b > a$ ) is given by

$$\text{Area} = \int_a^b y dx = \int_a^b f(x) dx$$

- Area of the region bounded by the curve  $x = g(y)$ ,  $y$ -axis and the lines  $y = c$  and  $y = d$  (where  $d > c$ ) is given by

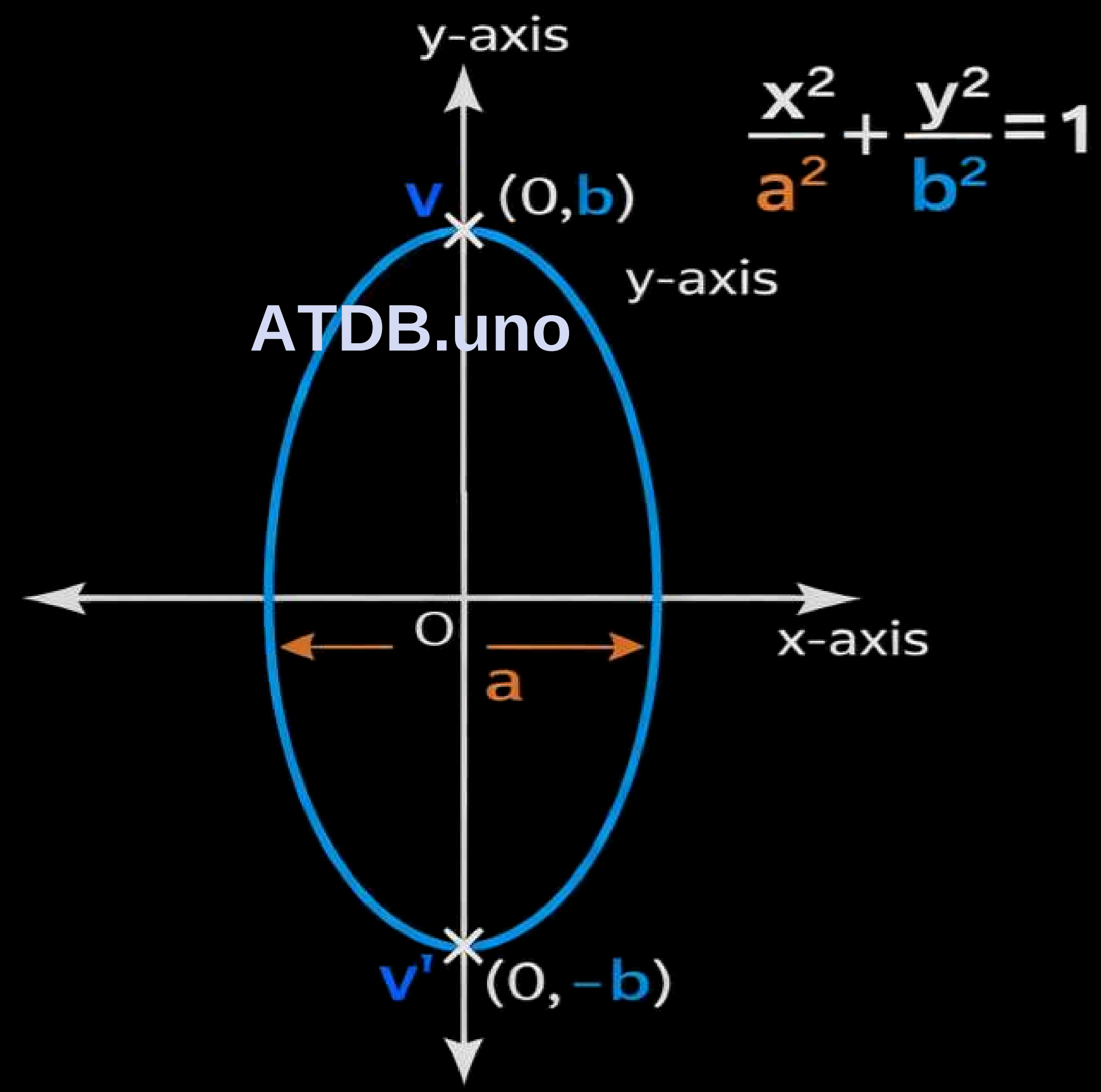
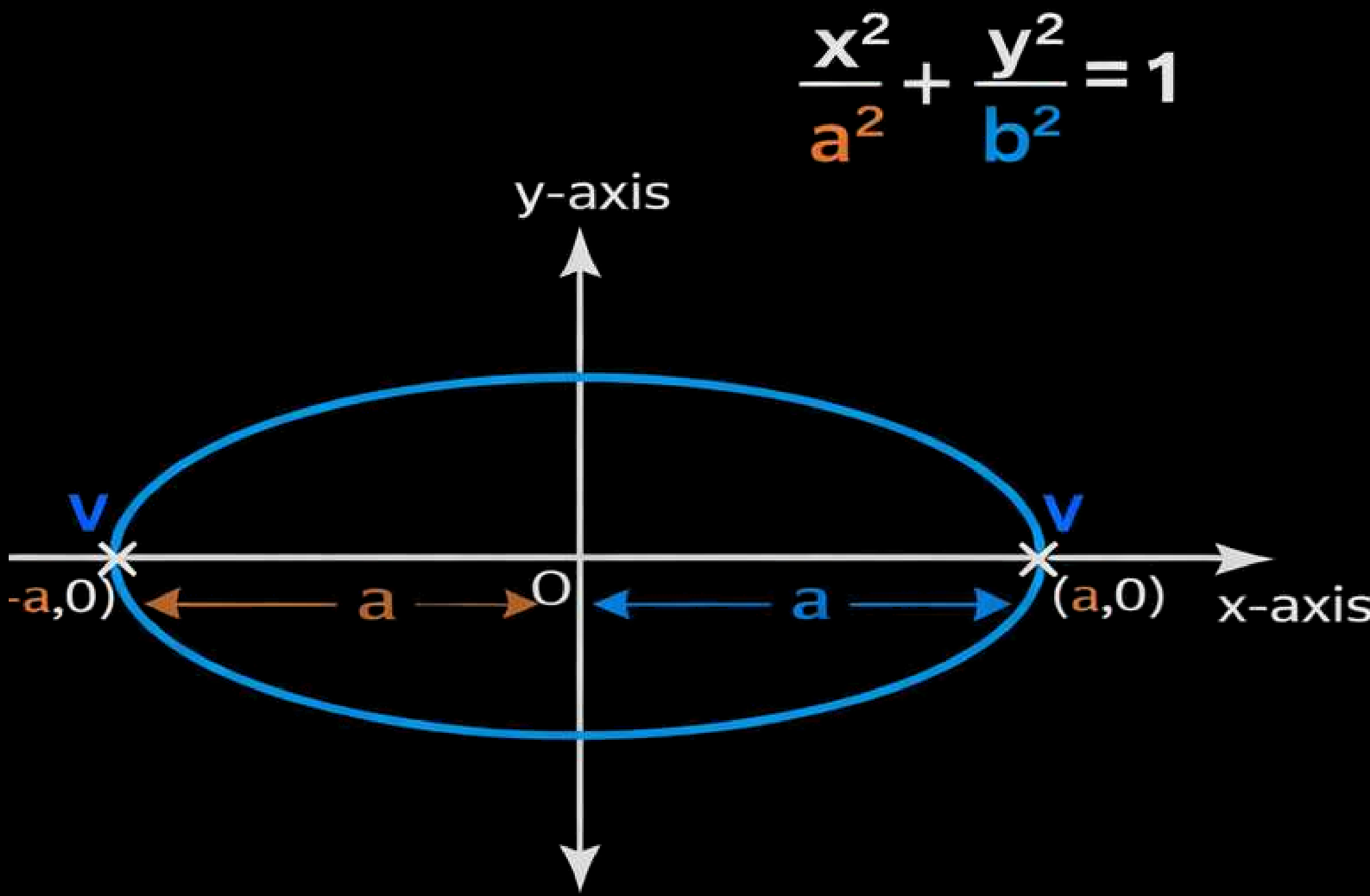
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$$\text{Area} = \int_c^d x dy = \int_c^d g(y) dy$$



# Application of integrals

## Ellipse:



# Differential Equations



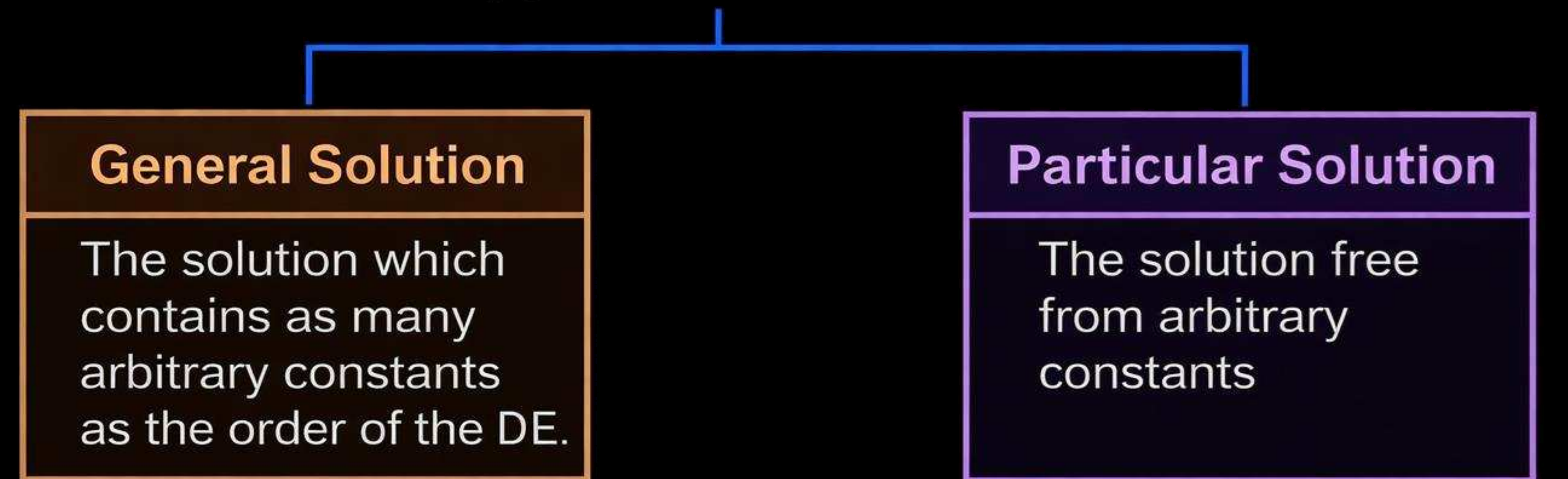
## KEY POINTS :

- **DIFFERENTIAL EQUATION** : is an equation involving derivatives of the dependent variable w.r.t independent variables and the variables themselves.
- **ORDINARY DIFFERENTIAL EQUATION (ODE)** : A.D.E. involving derivatives of the dependent variable w.r.t only one independent variable is an ordinary D.E.  
In class XII ODE is referred to as D.E. ATDB.uno
- **PARTIAL DIFFERENTIAL EQUATION (PDE)** : A.D.E involving derivatives w.r.t more than one independent variables is called a partial D.E.
- **ORDER of a DE** : is the order of the highest order derivative occurring in the D.E.
- **DEGREE of a D.E.** : is the highest power of the highest order derivative occurring in the D.E provided D.E is a polynomial equation in its derivatives. It is always a whole no.
- **SOLUTION OF THE D.E** : A relation between involved variables, which satisfy the given D.E is called its solution.

# Differential Equations



## Two Types of Solutions of DE



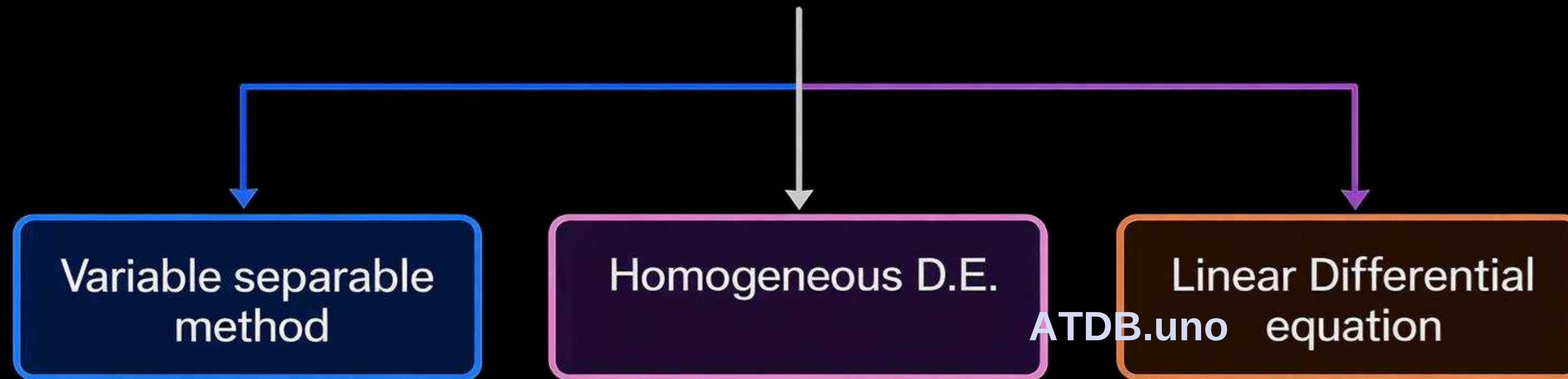
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- **Formation of a differential equation:** We differentiate the function successively as many times as the arbitrary constants in the given function and then eliminate the arbitrary constants from these equations.
- **ORDER of A D.E :** Is equal to the number of arbitrary constants in the general solution of a D.E.

# Differential Equations



Solution of a First Order First Degree D.E.



- **"VARIABLE SEPARABLE METHOD"**: is used to solve D.E. in which variables can be separated completely i.e, terms containing  $x$  should remain with  $dx$  and terms containing  $y$  should remain with  $dy$ .



# Differential Equations

- **"Homogeneous differential equation :** D.E. of the form  $\frac{dy}{dx} = F(x, y)$  where  $F(x, y)$  is a homogeneous function of degree 0

i.e.  $F(\lambda x, \lambda y) = \lambda^0 F(x, y)$

or  $F(\lambda x, \lambda y) = F(x, y)$  for some non-zero constant  $\lambda$ .

To solve this type put  $y = vx$

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To Solve homogenous D.E of the type  $\frac{dx}{dy} = G(x, y)$ , we make substitution  $x = vy$

# Differential Equations



- **Linear differential equation :** A.D.E of the form  $\frac{dy}{dx} + Py = Q$  where  $P$  and  $Q$  are constants or functions of  $x$  only is known as first order linear differential equation.

Its solution

$$y \cdot (I.F) = \int Q \times (I.F) dx + C, \text{ where}$$

$$I. F = \text{Integrating factor} = e^{\int P dx} \quad \text{ATDB.uno}$$

Another form of Linear Differential Equation is  $\frac{dx}{dy} + P_1 x = Q_1$ , where  $P_1$  and  $Q_1$  are constants or functions of  $y$  only.

Its solution is given as

$$x \cdot (I. F) = \int Q_1 X(I. F) dy + C, \text{ where } I. F. = e^{\int P_1 dy}$$

# Vectors



A quantity that has magnitude as well as direction is **called a vector**.

It is denoted by a directed line segment.

Position vector of a point  $P(a, b, c)$  w.r.t. origin  $(0, 0, 0)$  is denoted by  $\overrightarrow{OP}$  where  $\overrightarrow{OP} = a\hat{i} + b\hat{j} + c\hat{k}$  and  $|\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2}$ .

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**Remark :** The vectors defined above are such that any of them may be subject to its parallel displacement without changing its magnitude and direction. Such vectors are **called free vectors**. Throughout this chapter, we will be dealing with free vectors only.

# Vectors



**Zero Vector:** A vector whose initial and terminal points coincide, is called a zero vector (or null vector), and denoted as  $\vec{0}$ . Zero vector can not be assigned a definite direction as it has zero magnitude. Or, alternatively otherwise, it may be regarded as having any direction. The vectors  $\overrightarrow{AA}$ ,  $\overrightarrow{BB}$  represent the zero vector,

**Unit Vector :** A vector whose magnitude is unity (i.e., 1 unit) is called a unit vector. The unit vector in the direction of a given vector  $\vec{a}$  is denoted by  $\hat{a}$ .

**Coinitial Vectors :** Two or more vectors having the same initial point are called coinital vectors.

**Collinear Vectors** Two or more vectors are said to be collinear if they are parallel to the same line, irrespective of their magnitudes and directions.



# Vectors

**Equal Vectors :** Two vectors  $\vec{a}$  and  $\vec{b}$  are said to be equal, if they have the same magnitude and direction regardless of the positions of their initial points, and written as  $\vec{a} = \vec{b}$ .

**Negative of a Vector :** A vector whose magnitude is the same as that of a given vector (say,  $\overrightarrow{AB}$ ), but direction is opposite to that of it, is called negative of the given vector. **For example**, vector  $\overrightarrow{BA}$  is negative of the vector  $\overrightarrow{AB}$ , and written as  $\overrightarrow{BA} = -\overrightarrow{AB}$ .

# Vectors



- If  $\vec{a}$  and  $\vec{b}$  are two collinear vectors, then  $\vec{a} = \lambda \vec{b}$  where  $\lambda$  is some non-zero scalar.
- The scalar components of a vector are its direction ratios, and represent its projections along the respective axes.
- The magnitude ( $r$ ), direction ratios ( $a, b, c$ ) and direction cosines ( $l, m, n$ ) of any vector are related as:

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$$l = \frac{a}{r}, m = \frac{b}{r}, n = \frac{c}{r}$$



# Vectors

- If two vectors  $\vec{a}$  and  $\vec{b}$  are represented in magnitude and direction by the two sides of a triangle in order, then their sum  $\vec{a} + \vec{b}$  is represented in magnitude and direction by third side of a triangle taken in opposite order. This is called triangle law of addition of vectors.
- The vector sum of two coinitial vectors is given by the diagonal of the parallelogram whose adjacent sides are the given vectors. **ATDB.uno**
- The multiplication of a given vector by a scalar  $\lambda$ , changes the magnitude of the vector by the multiple  $|\lambda|$ , and keeps the direction same (or makes it opposite) according as the value of  $\lambda$  is positive (or negative).
- For a given vector  $\vec{a}$ , the vector  $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$  gives the unit vector in the direction of  $\vec{a}$ .



# Vectors

- The position vector of a point R dividing a line segment joining the points P and Q whose position vectors are  $\vec{a}$  and  $\vec{b}$  respectively, in the ratio  $m : n$ 
  - (i) Internally, is given by  $\frac{n\vec{a} + m\vec{b}}{m + n}$ .
  - (ii) Externally, is given by  $\frac{m\vec{b} - n\vec{a}}{m - n}$ .

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# Vectors

## PRODUCT OF TWO VECTORS

There are two types of products between two vectors.

1. Scalar Product (OR) Dot Product.
2. Vector Product (OR) Cross Product.

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# Vectors



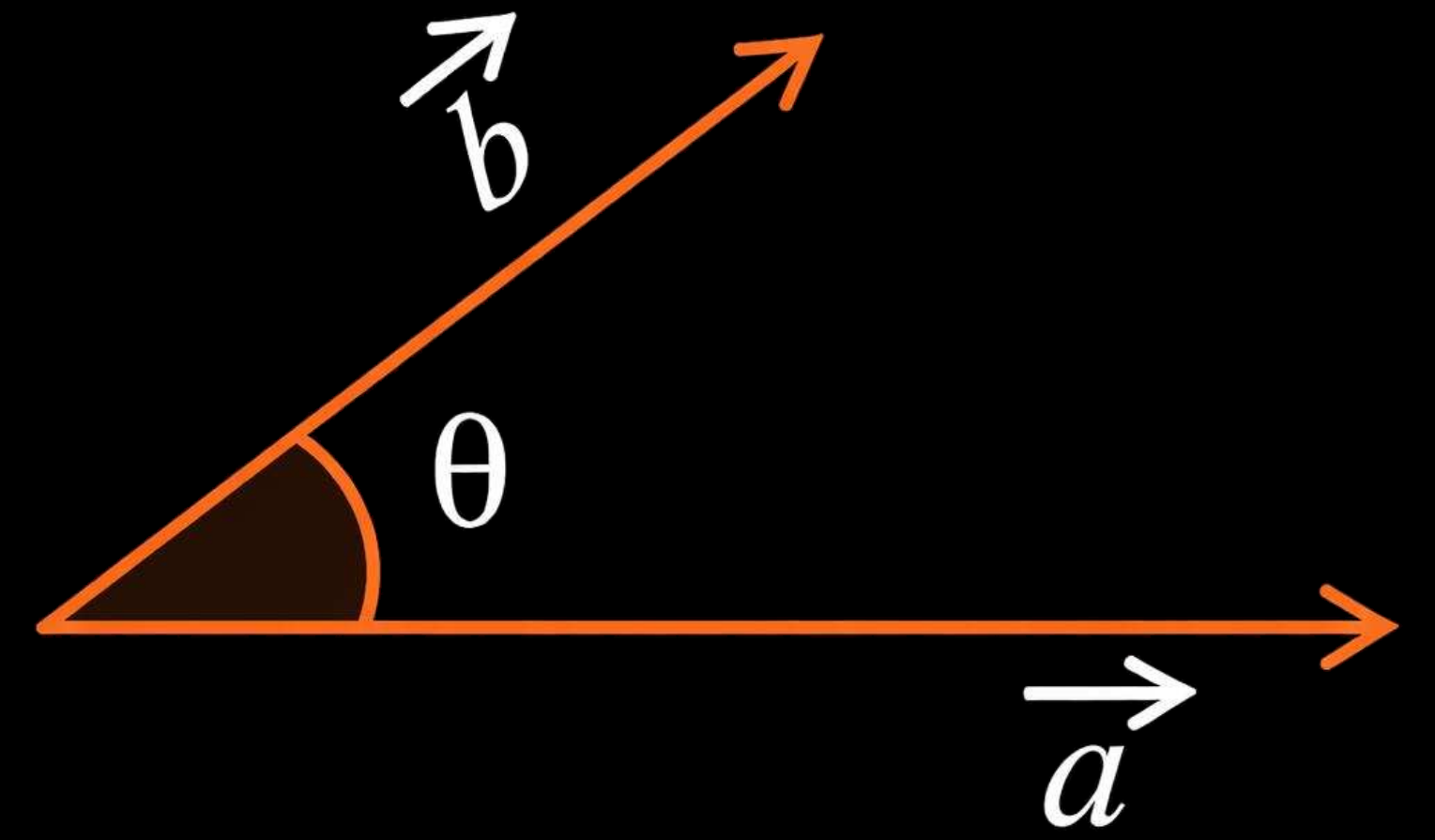
**Definition 2 :** The scalar product of two nonzero vectors  $\vec{a}$  and  $\vec{b}$ , denoted by  $\vec{a} \cdot \vec{b}$ , is

defined as  $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos \theta$

where,  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$ ,  $0 \leq \theta \leq \pi$  (Figure).

If either  $\vec{a} = 0$  or  $\vec{b} = 0$  then  $\theta$  is not defined, and in this case, we define  $\vec{a} \cdot \vec{b} = 0$

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# Vectors

## Observations

1.  $\vec{a} \cdot \vec{b}$  is a real number.
2. Let  $\vec{a}$  and  $\vec{b}$  be two nonzero vectors, then  $\vec{a} \cdot \vec{b} = 0$  if and only if  $\vec{a}$  and  $\vec{b}$  are perpendicular to each other. i.e.

$$\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$$

3. If  $\theta = 0$ , then  $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$  ATDB.uno

In particular,  $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ , as  $\theta$  in this case is 0.

4. If  $\theta = \pi$ , then  $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$

In particular,  $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$ , as  $\theta$  in this case is  $\pi$ .

5. In view of the Observations 2 and 3, for mutually perpendicular unit vectors  $\hat{i}, \hat{j}$  and  $\hat{k}$ , we have

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1,$$

$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$$



# Vectors

6. The angle between  $\vec{a}$  and  $\vec{b}$  is given by

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}, \quad \theta = \cos^{-1} \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right)$$

7. The scalar product is commutative. i.e.

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

## Projection of a Vector:

Let  $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$ .

Let  $\theta$  be the angle between  $\vec{a}$  and  $\vec{b}$ .

$\therefore$  The Projection of  $\vec{b}$  on  $\vec{a} = OM = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$ ,

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Similarly the projection of  $\vec{a}$  on  $\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

# Vectors

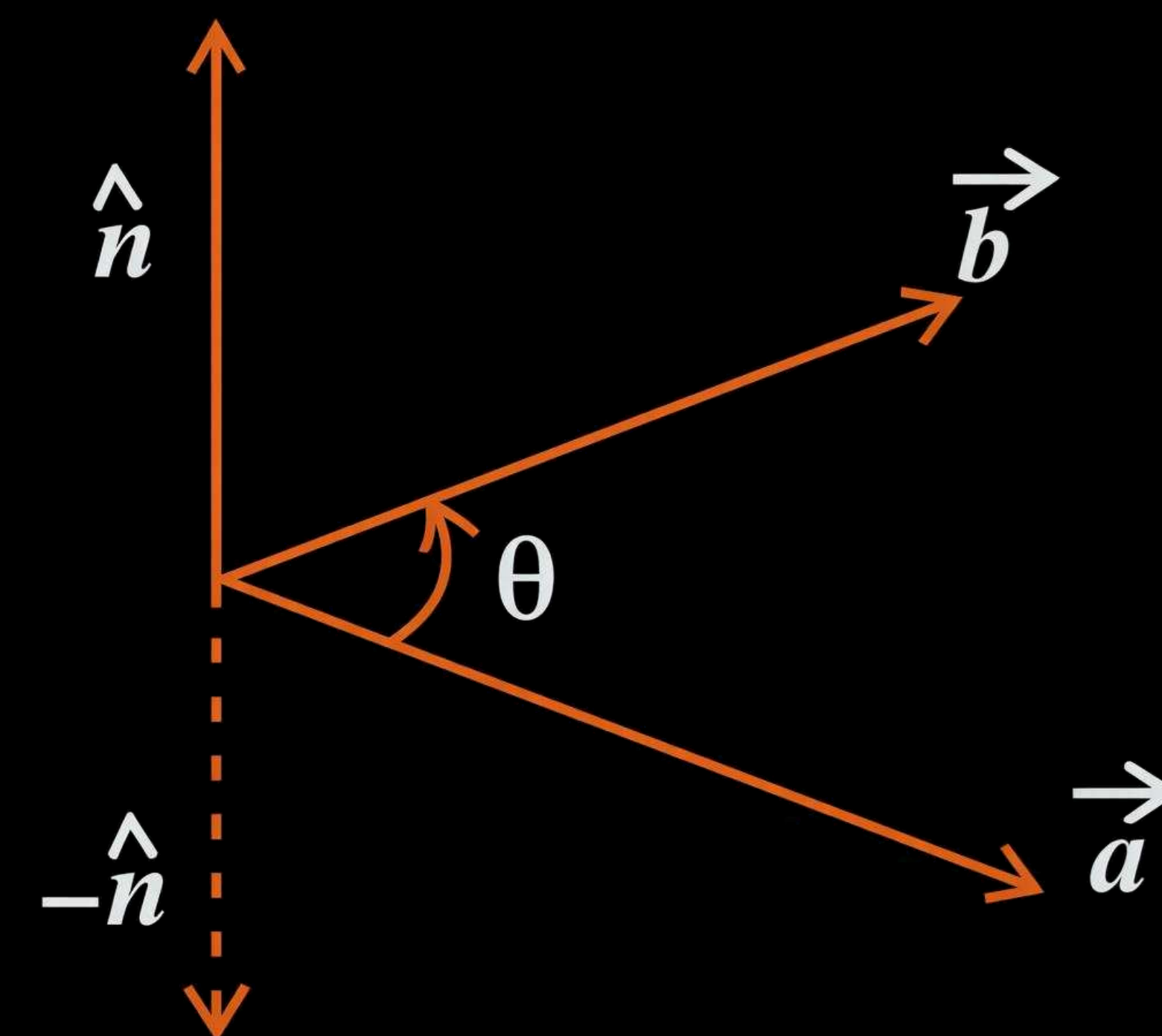


**Definition :** The vector product of two nonzero vectors  $\vec{a}$  and  $\vec{b}$ , is denoted by  $\vec{a} \times \vec{b}$  and defined as

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin \theta \hat{n}$$

where,  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$ ,  $0 \leq \theta \leq \pi$  and  $\hat{n}$  is a unit vector perpendicular to both  $\vec{a}$  and  $\vec{b}$ , such that  $\vec{a}$ ,  $\vec{b}$  and  $\hat{n}$  form a right handed system. i.e., the right handed system rotated from  $\vec{a}$  to  $\vec{b}$  moves in the direction of  $\hat{n}$ .

If either  $\vec{a} = \vec{0}$  or  $\vec{b} = \vec{0}$ , then  $\theta$  is not defined and in this case, we define  $\vec{a} \times \vec{b} = \vec{0}$ .



# Vectors

## Observations

1.  $\vec{a} \times \vec{b}$  is a vector.
2. Let  $\vec{a}$  and  $\vec{b}$  be two nonzero vectors. Then  $\vec{a} \times \vec{b} = \vec{0}$  if and only if  $\vec{a}$  and  $\vec{b}$  are parallel (or collinear) to each other, i.e.,

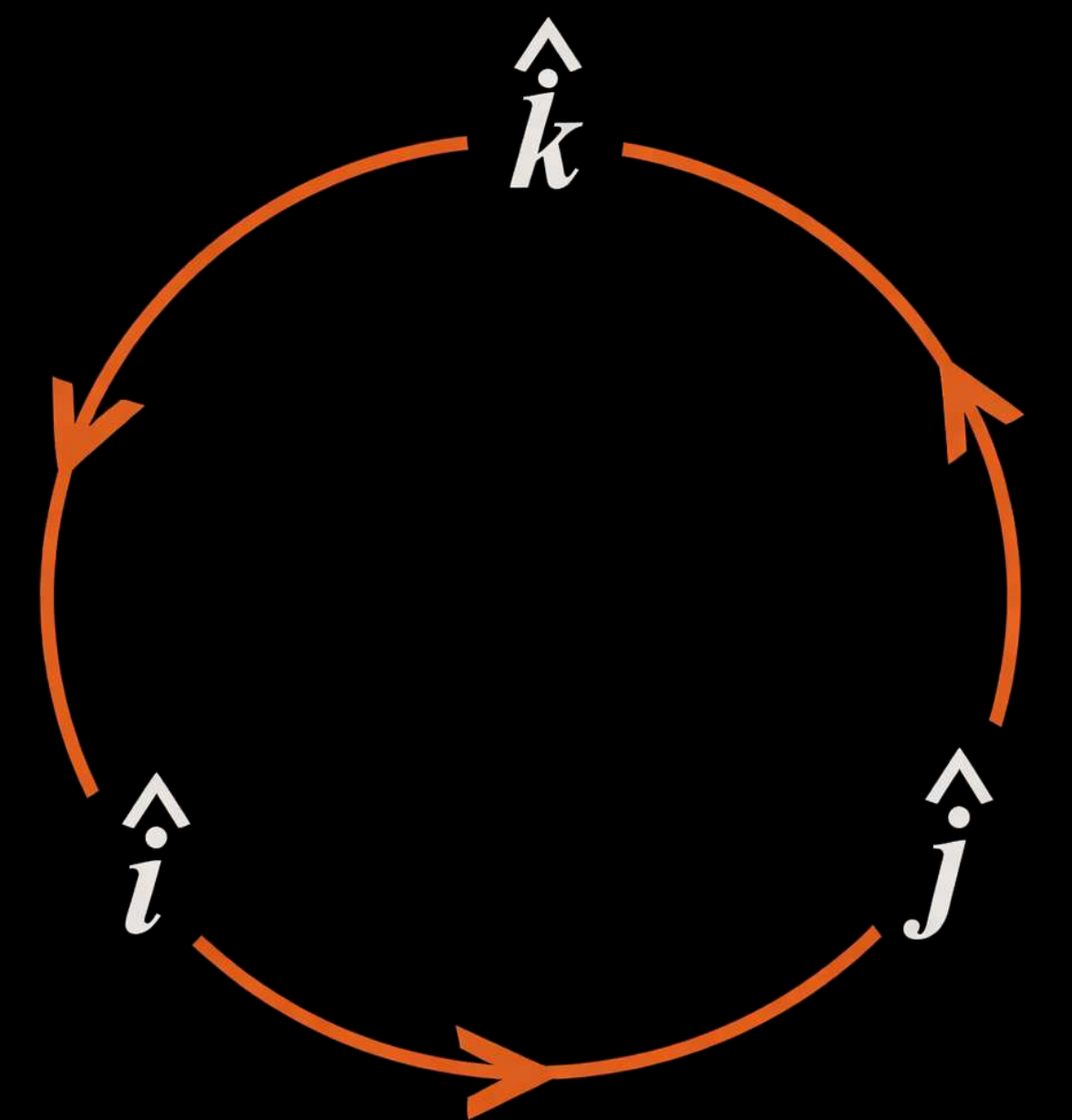
$$\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \parallel \vec{b}$$

In particular,  $\vec{a} \times \vec{a} = \vec{0}$  and  $\vec{a} \times (-\vec{a}) = \vec{0}$ , since in the first situation,  $\theta = 0$  and in the second one,  $\theta = \pi$ , making the value of  $\sin \theta$  to be 0.

3. If  $\theta = \frac{\pi}{2}$  then  $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|$ .
4. In view of the Observations 2 and 3, for mutually perpendicular unit vectors  $\hat{i}, \hat{j}$  and  $\hat{k}$ , we have

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

$$\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$$





# Vectors

5. In terms of vector product, the angle between two vectors  $\vec{a}$  and  $\vec{b}$  may be given as

$$\sin \theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| |\vec{b}|}$$

6. It is always true that the vector product is not commutative, as  $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ .

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# Vectors



Unit vector  $\perp$  both the vectors  $\vec{a}$  and  $\vec{b}$ :

$$\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}.$$

$$(1) \quad \text{then } |\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$$

If  $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$  and  $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$  are two vectors then the Vector Product  $\vec{a} \times \vec{b}$  is

$$\text{defined as } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

# 3D geometry



## Area of a Parallelogram :

- **Distance Formula :** Distance ( $d$ ) between two points  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- Direction ratio of a line through  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  are  $x_2 - x_1, y_2 - y_1, z_2 - z_1$
- Direction cosines of a line having direction ratios as  $a, b, c$  are:

$$l = \pm \frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \pm \frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

# 3D geometry



## Equation of line in space:

|      | Vector form                                                                                              |      | Cartesian form                                                                                                                                           |
|------|----------------------------------------------------------------------------------------------------------|------|----------------------------------------------------------------------------------------------------------------------------------------------------------|
| (i)  | Passing through point $\vec{a}$ and parallel to vector $\vec{b}$ ; $\vec{r} = \vec{a} + \lambda \vec{b}$ | (i)  | Passing through point $(x_1, y_1, z_1)$ and having direction ratios $a, b, c$                                                                            |
| (ii) | Passing through two points $\vec{a}$ and $\vec{b}$ ; $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$    | (ii) | Passing through two points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ ; $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$ |

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$



# 3D geometry

## Vector form

(i) For lines  $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$  and  $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ ,  $\cos \theta = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|}$

where 'θ' is the angle between two lines.

(iii) Lines are perpendicular if  $\vec{b}_1 \cdot \vec{b}_2 = 0$

(iv) Lines are parallel if  $\vec{b}_1 = k \vec{b}_2$ ; ATDB.uno

$k \neq 0$

(II) Lines are perpendicular If

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

(i) Lines are parallel if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

# 3D geometry



## The shortest distance between two skew lines

$$\vec{r} = \vec{a}_1 + \lambda \vec{b}_1 \text{ and } \vec{r} = \vec{a}_2 + \mu \vec{b}_2 \text{ is } d = \left| \frac{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

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# LPP



## KEY POINTS :

- **Optimisation problem** : is a problem which seeks to maximize or minimize a function. An optimisation problem may involve maximization of profit, minimization of transportation cost etc, from available resources.
- **A Linear programming problem (LPP)** : LPP deals with the optimization (maximization / minimization) of a linear function of two variables (say  $x$  and  $y$ ) known as objective function subject to the conditions that the variables are non negative and satisfy a set of linear inequalities (called linear constraints). A LPP is a special type of optimisation problem.



# LPP

- **Objective function :** Linear function  $z = ax + by$  where  $a$  and  $b$  are constants which has to be maximised or minimised is called a linear objective function.
- **Decision variables :** In the objective function  $z = ax + by$ ,  $x$  and  $y$  are called decision variables.
- **Constraints :** The linear inequalities or restrictions on the variables of an LPP are called constraints. The conditions  $x \geq 0, y \geq 0$  are called non-negative constraints.

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- **Feasible Region :** The common region determined by all the constraints including non-negative constraints  $x \geq 0, y \geq 0$  of a LPP is called the feasible region for the problem.
- **Feasible Solution :** Points within and on the boundary of the feasible region for a LPP represent feasible solutions.
- **Infeasible Solutions :** Any point outside the feasible region is called an infeasible solution.
- **Optimal (Feasible) Solution :** Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an optimal solution.
- **Theorem 1 :** Let R be the feasible region (convex polygon) for a LPP and let  $z = ax + by$  be the objective function. When z has an optimal value (maximum or minimum), where x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.

# LPP



- **Theorem 2 :** Let  $R$  be the feasible region for a LPP. & let  $z = ax + by$  be the objective function. If  $R$  is bounded, then the objective function  $z$  has both a maximum and a minimum value on  $R$  and each of these occur at a corner point of  $R$ .

If the feasible region  $R$  is unbounded, then a maximum or minimum value of the objective function may or not exist. However, if it exists it must occur at a corner point of  $R$ .

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# Probability



**Conditional Probability :** If A and B are two events associated with the same sample space of a random experiment, then the conditional probability of the event A under the condition that the event B has already occurred, written as  $P(A | B)$ , is given by

$$P(A | B) = \frac{P(A \cap B)}{P(B)}, P(B) \neq 0$$

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## Properties:

$$P(S | F) = P(F | F) = 1 \quad \dots(1)$$

$$P((A \cup B) | F) = P(A | F) + P(B | F) - P((A \cap B) | F) \quad \dots(2)$$

$$P(E' | F) = 1 - P(E | F) \quad \dots(3)$$

# Probability



**Multiplication Rule :** Let E and F be two events associated with a sample place of an experiment. Then

$$\begin{aligned} P(E \cap F) &= P(E)P(F | E) \quad P(E) \neq 0 \\ &= P(F)P(E | F) \quad P(F) \neq 0 \end{aligned}$$

If E, F, G are three events associated with a sample space, then

$$P(E \cap F \cap G) = P(E)P(F | E)P(G | (E \cap F)) \quad \text{ATDB.uno}$$

# Probability



**Independent Events :** Let E and F be two events, then if probability of one of them is not affected by the occurrence of the other, then E and F are said to be independent, i.e.,

$$P(F | E) = P(F), P(E) \neq 0$$

or

$$P(E | F) = P(E), P(F) \neq 0$$

or

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$$P(E \cap F) = P(E)P(F)$$

Three events A, B, C are mutually independent if

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

$$P(A \cap B) = P(A)P(B)$$

$$P(B \cap C) = P(B)P(C) \text{ and}$$

$$P(A \cap C) = P(A)P(C)$$

# Probability



**Theorem of Total Probability :** Let  $\{E_1, E_2, \dots, E_n\}$  be a partition of the sample space  $S$ . Let  $A$  be the any event associated with  $S$ , then  $P(A) = \sum_{j=1}^n P(E_j)P(A | E_j)$

**Baye's Theorem :** If  $E_1, E_2, \dots, E_n$  are mutually exclusive and exhaustive events associated with a sample space  $S$ , and  $A$  is any event associated with  $E_i$  's having non-zero probability, then

$$P(E_i | A) = \frac{P(A|E_i)P(E_i)}{\sum_{i=1}^n P(A|E_i)P(E_i)}$$



**Thank**  
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*You*