Centre of Mass & System of Particles.

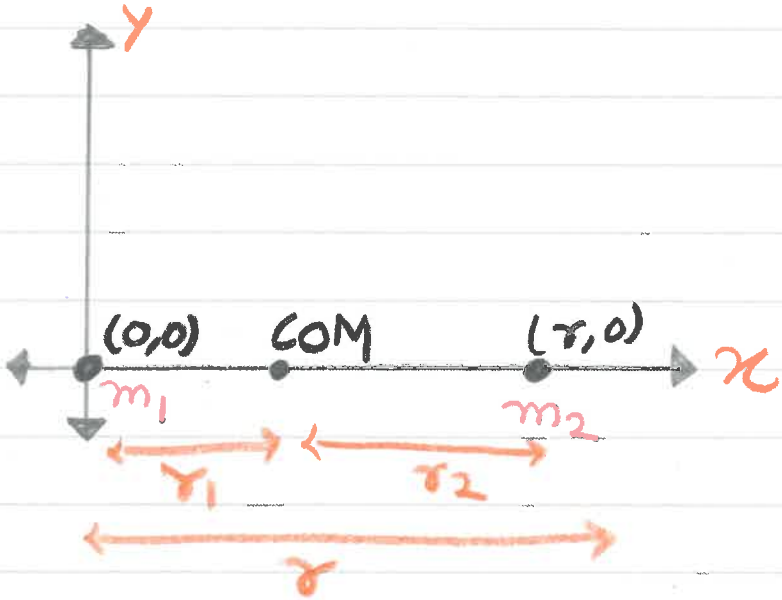
The centre of mass of an object/discrete particles is a unique point where all the mass can be assumed to be concentrated.

For a system of n point masses $m_1, m_2, m_3 \dots m_n$ whose position vector are given by $\vec{r_1}, \vec{r_2}, \vec{r_3} \dots \vec{r_n}$ respectively, the position vector of the centre of mass of the system is given as follows

$$\vec{r}_{com} = \frac{m_1\vec{r_1} + m_2\vec{r_2} \dots + m_n\vec{r_n}}{m_1 + m_2 \dots + m_n} = \frac{\sum_1^n m_i\vec{r_i}}{\sum_1^n m_i}$$

Thus the centre of mass is the weighted average of all the masses.

Centre of mass of two particles



Consider two particles of masses m_1 and m_2 separated by a distance r

Let us choose the origin at particle m_1.

Let the centre of mass be situated at a distance r_1 from m_1 and r_2 from m_2 as shown in the figure

Then

$$r_1 = \frac{m_1 \times 0 + m_2 \times r}{m_1 + m_2} = \frac{m_2 r}{m_1 + m_2} \quad \text{--- (i)}$$

And

$$r_2 = r - r_1 = r - \frac{m_2 r}{m_1 + m_2} = \frac{m_1 r}{m_1 + m_2} \quad \text{--- (ii)}$$

Dividing equation (i) by equation (ii)

$$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

The centre of mass divides the line joining the two particles in an inverse ratio of their masses

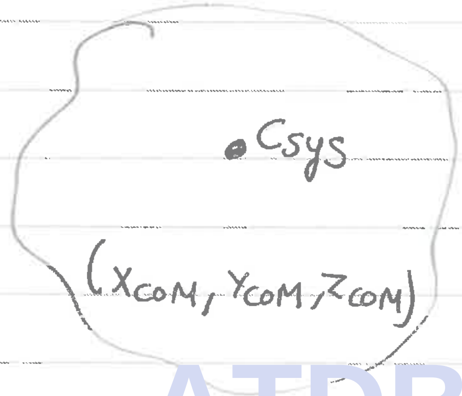
Centre of Mass of Symmetric Bodies.

- The centre of mass of continuous bodies are found using integration
- The centre of mass can be located outside

the physical body.

- If an object has uniform mass distribution and the geometry is symmetrical, then the centre of mass lies at the centre of symmetry.

Centre of Mass of Continuous Body.



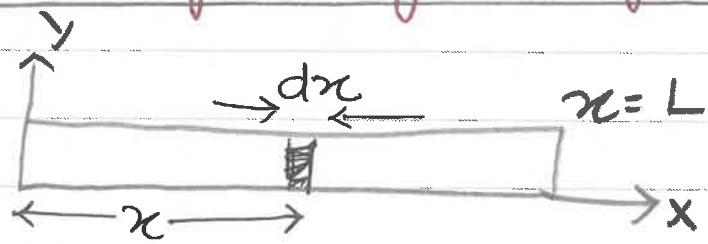
- Let us consider a body having a continuous distribution of matter, then the centre of mass of such a body is given as follows

$$x_{com} = \frac{\int x\,dm}{\int dm}$$

$$y_{com} = \frac{\int y\,dm}{\int dm}$$

$$z_{com} = \frac{\int z\,dm}{\int dm}.$$

Centre of Mass of a Uniform rod.



Consider a rod of Mass M and length L lying along the x-axis with one end at the origin

For finding the centre of mass of this rod, let us ~~cond~~ consider a small element of length dx and Mass dm at a distance x from the origin along the x-axis as shown in figure

Now by unitary method we get

$$dm = \frac{M}{L} dx$$

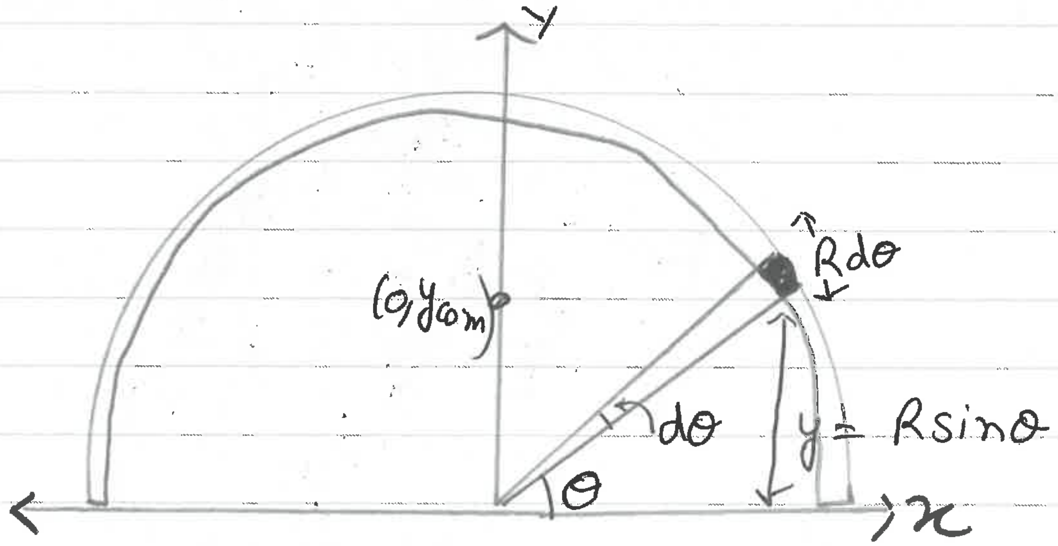
The centre of mass is given as follows

$$x_{COM} = \frac{\int x\,dm}{\int dm}$$

$$x_{COM} = \int_0^L \frac{M}{L} x\,dx = \frac{1}{L}\left[\frac{x^2}{2}\right]_0^L$$

$$x_{com} = \frac{L}{2}.$$

Centre of Mass of a semi-circular Ring.



~~Cond~~ consider a uniform semi-circular ring of radius R and Mass M as shown in the figure

As the ring is symmetrical about the y-axis thus the centre of mass will lie along the y-axis.

Let the coordinate of the centre of mass be $(0, y_{COM})$.

Consider a small element of length $dl = Rd\theta$ mass dm, subtending angle $d\theta$ at the centre, and located at an angle θ, with the positive x-axis as shown in the figure

Now, mass of the element is as follows

$$dm = \frac{M}{\pi R} \times (Rd\theta) = \frac{M}{\pi} d\theta$$

And $y = R\sin\theta$

The centre of mass is given as follows:

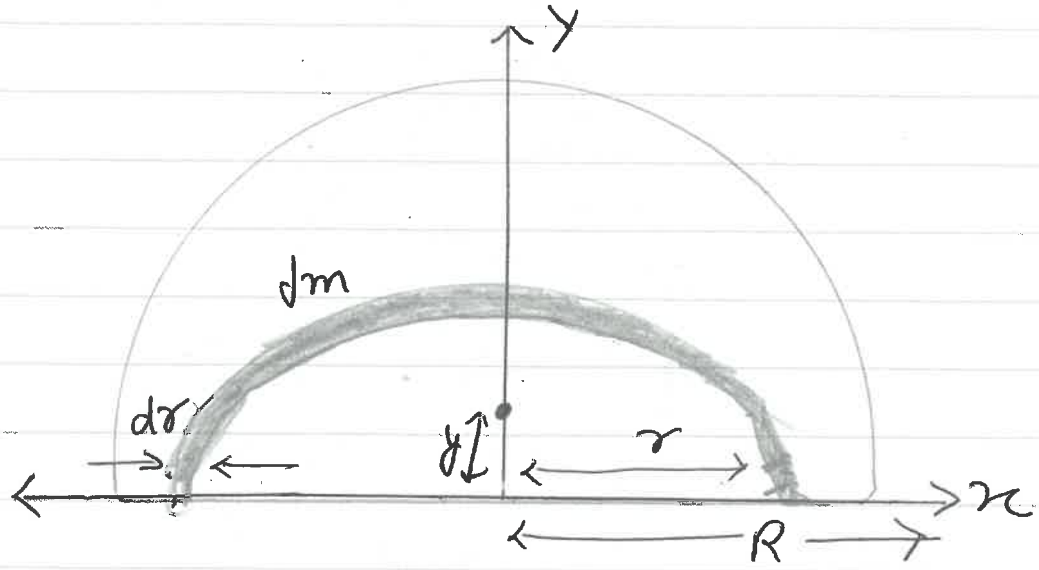
$$y_{com} = \frac{\int y\,dm}{\int dm}$$

$$y_{com} = \frac{\int_0^{\pi} R\sin\theta\left(\frac{M}{\pi}\right)d\theta}{\int_0^{\pi}\left(\frac{M}{\pi}\right)d\theta} = \frac{-\frac{MR}{\pi}[\cos\theta]_0^{\pi}}{M}$$

$$y_{com} = \frac{2R}{\pi}$$

Therefore, the coordinates of centre of mass are $(0, y_{com}) = \left(0, \frac{2R}{\pi}\right)$

Centre of Mass of a Semicircular Disc.



Consider a uniform semicircular disc of Radius R and M mass as shown in figure. As the disc is symmetrical about the y-axis, the centre of mass will lie along the y-axis.

Let ~~us~~ the coordinate of the centre of mass be $(0, y_{com})$. Consider a thin circular element of thickness dr and mass dm at radius r from the centre as shown in the figure.

Now, the area of the element is.

$$dA = \frac{1}{2}\pi\{(r+dr)^2 - r^2\}$$

$$\therefore dA = \frac{1}{2}\pi\{r^2 + (dr)^2 + 2 \times r \times dr - r^2\}$$

Ignoring $(dr)^2$ as it is very small we get

$$dA = \pi r dr$$

Mass of the element is

$$dm = \frac{M}{A} \times dA = \frac{2M}{\pi R^2} \times \pi r \times dr = \frac{2M r \times dr}{R^2}$$

Now the y-coordinate of the centre of mass is given by.

$$y_{com} = \frac{1}{M}\int y\, dm$$

Here y = Centre of mass of the elemental ring

$$= \frac{2r}{\pi}$$

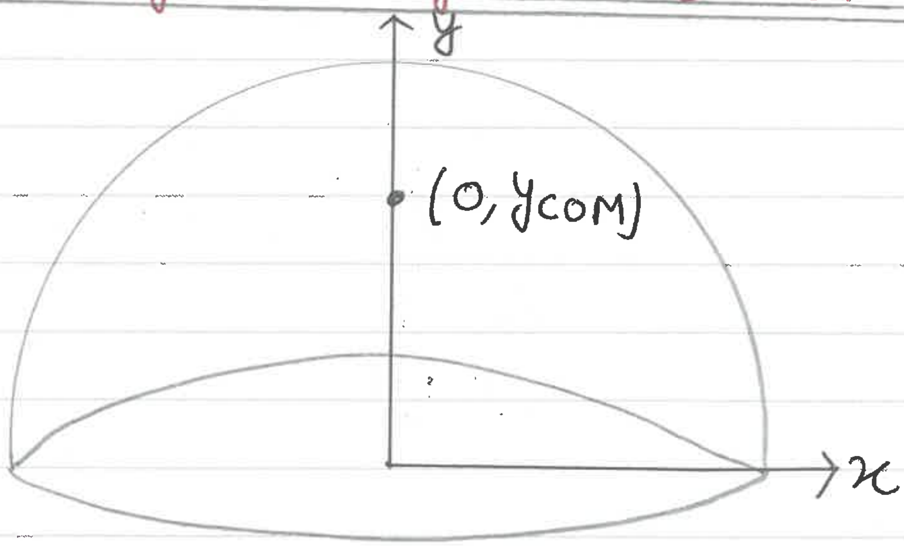
$$\therefore \quad y_{com} = \frac{1}{M}\int_0^R \left(\frac{2r}{\pi} \times \frac{2M}{R^2} \times r \times dr\right)$$

$$\Rightarrow \quad y_{COM} = \frac{4}{\pi R^2}\int_0^R r^2 dr = \frac{4}{\pi R^2} \times \left[\frac{r^3}{3}\right]_0^R$$

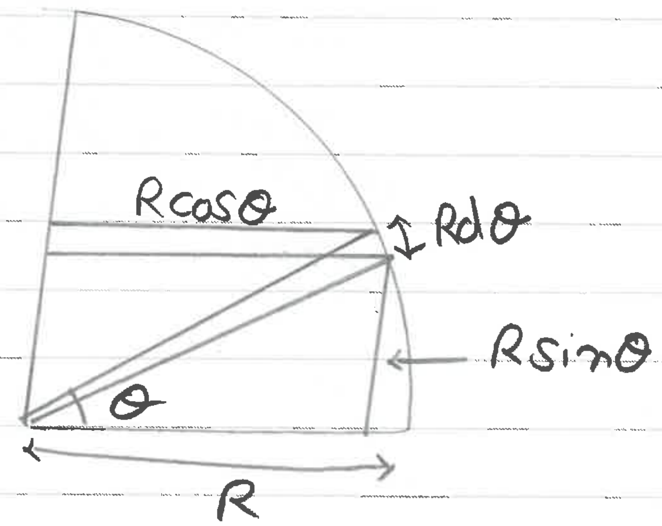
$$y_{COM} = \frac{4R}{3\pi}$$

Therefore the coordinate of the centre of mass are $\left(0, \frac{4R}{3\pi}\right)$

Centre of Mass of Hollow Hemisphere.



Consider a uniform hollow hemisphere of radius R and mass M as shown in the figure.



As the hemisphere is symmetrical about the y-axis, the centre of mass will lie along the y-axis.

Let the coordinates of the centre of mass is $(0, y_{com})$ The hollow hemisphere can be considered to be made up of many thin rings that are stacked one above the other and their radius decreases as we go from the base to the top.

Consider a thin ring of mass dm at an angle θ from the centre as shown in the figure.

Now an area of the element = Circumference × Thickness.

$$dA = 2\pi \times (R\cos\theta) \times Rd\theta.$$

$$dA = 2\pi R^2 \cos\theta d\theta$$

Surface density of the hemisphere is $\sigma = \frac{M}{2\pi R^2}$

Mass of the element is

$$dm = \sigma \times dA = \frac{M}{2\pi R^2} \times 2\pi R^2 \cos\theta d\theta = M\cos\theta d\theta$$

Now the y-coordinate of the centre of mass is given by,

$$y_{com} = \frac{1}{M}\int y\,dm$$

Here y = Centre of mass of the elemental ring

$= R\sin\theta$

$$y_{COM} = \frac{1}{M} \times \int_0^{\pi/2} R\sin\theta \times M \times \cos\theta d\theta$$

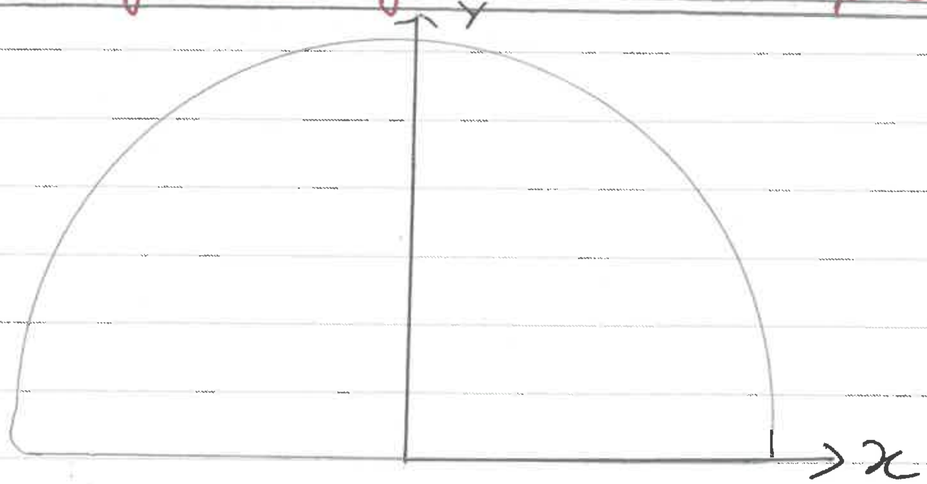
$$y_{com} = \frac{R}{2}\int_0^{\pi/2} \sin 2\theta\, d\theta$$

$$= \frac{R}{2}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$y_{COM} = \frac{R}{2}$$

Therefore, the coordinate of the centre of mass are $\left\{0, \frac{R}{2}\right\}$.

Centre of Mass of a solid hemisphere.

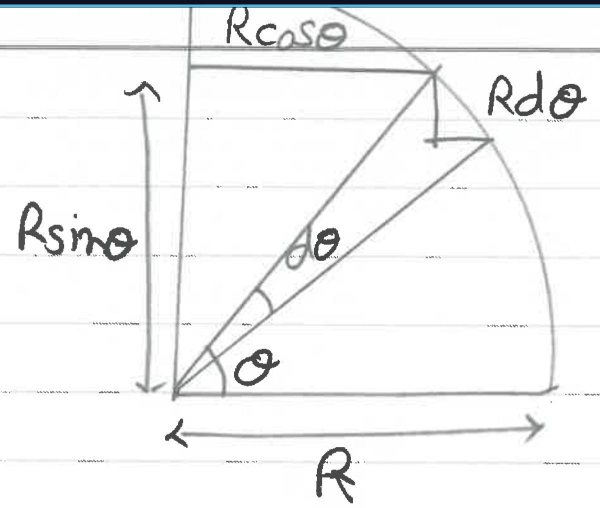


Consider a uniform solid hemisphere of Radius R and mass M as shown in the figure. As the hemisphere is symmetrical about the y axis, the centre of mass will be along the y-axis.

Let the coordinate of the centre of mass be $(0, y_{COM})$. The solid hemisphere can be consider to be made up of many thin discs that are stacked one above the other and their radius decreases as we go from the base to the top.

Consider a thin disc of mass dm at an angle θ from the centre as shown in the figure.

Now the volume of the element = Base area × height.



$$dv = \pi \times (R\cos\theta)^2 \times (R\cos\theta\, d\theta)$$

$$dv = \pi R^3 \cos^3\theta\, d\theta.$$

Volume density of the hemisphere is $\rho = \dfrac{M}{\frac{2\pi R^3}{3}} = \dfrac{3M}{2\pi R^3}$

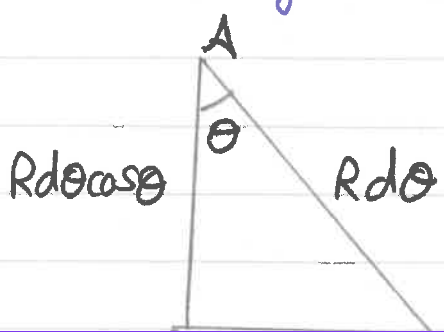
Mass of the element is:

$$dm = \rho \times dv = \frac{3M}{2\pi R^3} \times \pi R^3 \cos^3\theta\, d\theta = \frac{3M}{2}\cos^3\theta\, d\theta$$

Now the y-coordinate of the centre of mass is given by

$$y_{com} = \frac{1}{M}\int y\, dm$$

Here y = centre of mass of elemental disc $= R\sin\theta$



$$y_{com} = \frac{1}{M}\int_0^{\pi/2} R\sin\theta \times \frac{3M}{2} \times \cos^3\theta \, d\theta$$

$$\Rightarrow y_{com} = \frac{3R}{2} \times \int_0^{\pi/2} \sin\theta \cos^3\theta \, d\theta$$

$$y_{com} = \frac{3R}{2}\int_0^{\pi/2} \sin\theta \cos^3\theta \, d\theta.$$

$$\text{Let } I = \int_0^{\pi/2} \sin\theta \cos^3\theta \, d\theta \quad -(ii)$$

from properties of definite integration

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$\Rightarrow \int_0^{\pi/2} \sin\theta \cos^3\theta \, d\theta = \int_0^{\pi/2} \sin\left(\frac{\pi}{2}-\theta\right) \cos^3\left(\frac{\pi}{2}-\theta\right) d\theta$$

$$I = \int_0^{\pi/2} \cos\theta \sin^3\theta \, d\theta \quad --(iii)$$

Adding equation (ii) and equation (iii) we get

$$2I = \int_0^{\pi/2} \sin\theta \cos\theta \, (\sin^2\theta + \cos^2\theta)\,d\theta.$$

$$= \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta$$

$$I = \frac{1}{2}\int_0^{\pi/2} \sin\theta \cos\theta \, d\theta = \frac{1}{4}\int_0^{\pi/2} \sin 2\theta \, d\theta$$

$$I = \frac{1}{8}\left[-\cos 2\theta\right]_0^{\pi/2}$$

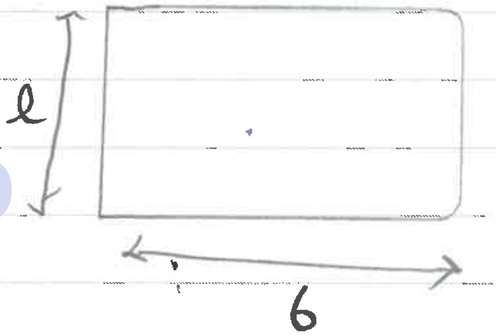
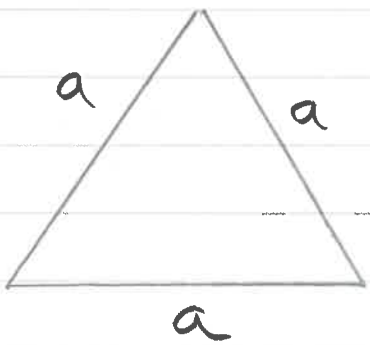
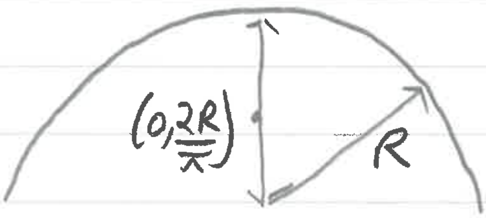
$$\Rightarrow I = \frac{1}{\cdot}$$

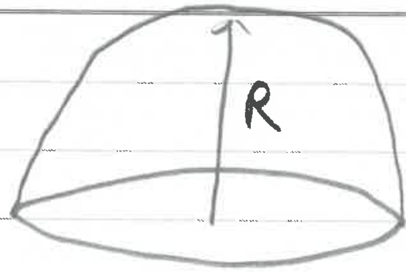
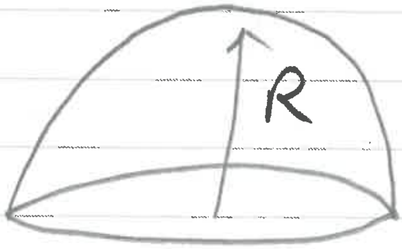
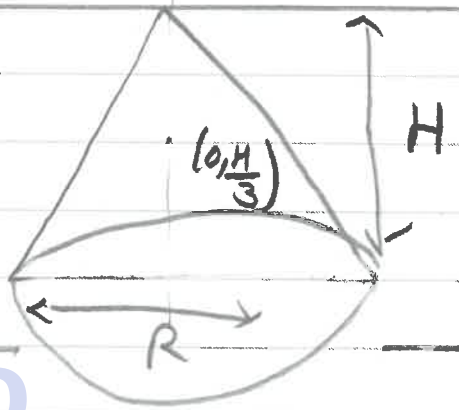
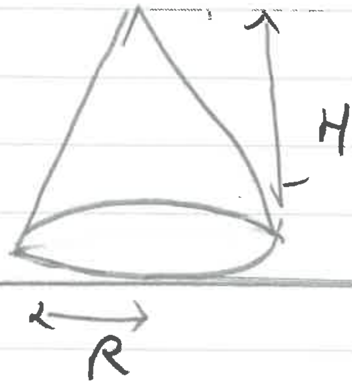
Put in equation (i) we get

$$y_{com} = \frac{3R}{8}$$

Therefore, the coordinates of the centre of mass ar $\left(0, \frac{3R}{8}\right)$

COM of certain shapes.

1)	Rectangular lamina	$\left(\frac{b}{2}, \frac{l}{2}\right)$	
2)	Triangular lamina	$\left(\frac{a}{2}, \frac{a}{2\sqrt{3}}\right)$	
3)	Semi-circular ring	$\left(0, \frac{2R}{\pi}\right)$	
4)	Semi-circular disc	$\left(0, \frac{4R}{3\pi}\right)$	

4)	Hollow hemisphere	$\left(0, \frac{R}{2}\right)$	
5)	Solid hemisphere	$\left(0, \frac{3R}{8}\right)$	
6)	Hollow cone	$\left(0, \frac{H}{3}\right)$	
7)	Solid cone	$\left(0, \frac{H}{4}\right)$	

Displacement of Centre of Mass.

For a system of n particles of masses m_1, m_2 --- m_n with position vector $\vec{r_1}, \vec{r_2}$ --- $\vec{r_n}$ respectively the position vector of the centre of mass is given as follows:

$$\vec{r}_{COM} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3} - - - m_n\vec{r_n}}{m_1 + m_2 + m_3 - - - m_n}$$

$$= \sum_{i=1}^{n} \frac{m_i\vec{r_i}}{\sum_1^n m_i}$$

When the system is moving, let $\vec{V}_{com}$ and $\vec{a}_{com}$ be the velocity and acceleration of the centre of mass of the system

Differentiating equation (i) with respect to time we get.

$$\frac{\partial \vec{r}_{COM}}{\partial t} = \frac{m_1\frac{\partial r_1}{\partial t} + m_2\frac{\partial r_2}{\partial t} - - - - - m_n\frac{\partial r_n}{\partial t}}{m_1 + m_2 - - - + m_n}$$

$$= \frac{\sum_1^n m_i \frac{d\vec{r_i}}{\partial t}}{\sum_1^n m_n}$$

$$V_{COM} = \frac{m_1 v_1 + m_2 V_2 - - - + m_n v_n}{m_1 + m_2 - - + m_n} = \frac{\sum_1^n m_i v_i}{\sum_1^n m_i}$$

Similarly

$$a_{COM} = \frac{\sum_1^n m_i a_i}{}$$

For a system of n particles of masses $m_1, m_2 \ldots m_n$ displacement by $\Delta r_1, \Delta r_2 \ldots \Delta \vec{r_n}$, respectively, the charge in position vector of the centre of mass is given as follows:-

$$\Delta r_{COM} = \frac{m_1 \Delta\vec{r_1} + m_2 \Delta\vec{r_2} \ldots + m_n \Delta\vec{r_n}}{m_1 + m_2 \ldots + m_n}$$

$$= \frac{\sum_{i=1}^{n} m_i \Delta\vec{r_i}}{\sum_{i=1}^{n} m_i}$$

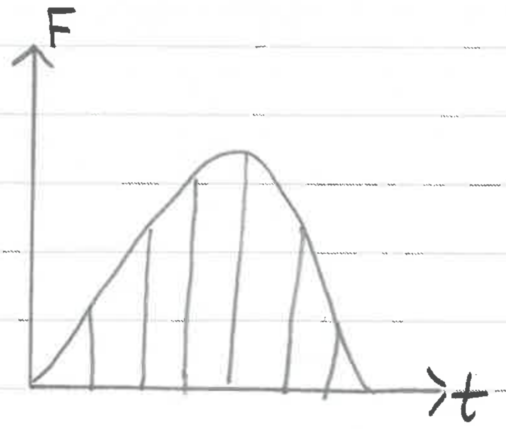
Impulse

The charge in the momentum suffered by a system due to action of a force (F) of an infinitesimal amount of time is known as impulse (J) given by the force to the system.

$$\vec{J} = \int \vec{F} dt = \Delta\vec{P} = \vec{P_f} - \vec{P_i}$$

The impulse applied to an object in a given interval of time is the area under the Force vs time (F-t) graph in the same time interval

$$\text{Area} = \vec{J} = \int \vec{F} dt = \Delta\vec{P} = \vec{P_f} - \vec{P_i}$$



* If any force can change the momentum appreciably even in a infinitesimal time, then it is an impulsive force Example Normal reaction, friction force, tension force etc.

Collisions

Collision happen when an impulsive force acts between two or more bodies for a short time which results in a change of their momenta.

Salient feature of collision:-

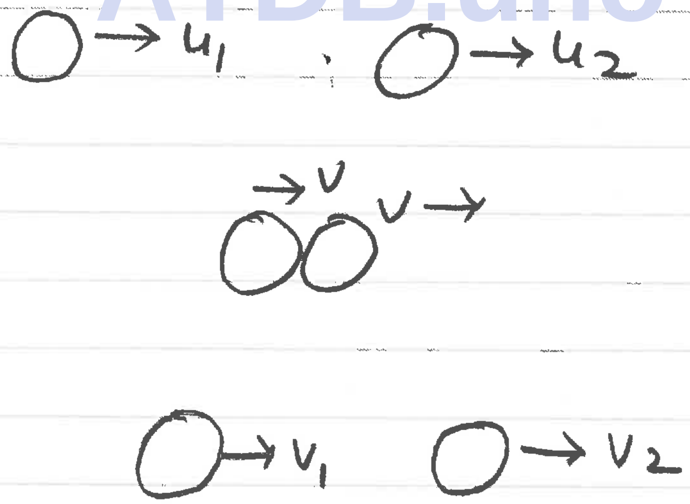
- The effect sof non-impulsive force like gravity are not taken into account.
- Impulsive force are fiction, tension and normal reaction
- Non impulsive force are gravity, Spring force and electrostatic force
- Particle may or may not be in physical contact.
- The time interval considered is small

Coefficient of restitution (e)

It is the ratio of the impulses of reformation (J_r) to deformation (J_d) of either body undergoing collision.

$$e = \frac{J_r}{J_d} = \frac{\int F_r\,dt}{\int F_d\,dt}$$

Coefficient of restitution signifies how much deformation has happened to the bodies due to collision. Its value can be maximum 1 and minimum 0



Before collision

During collision

After collision

Let two balls are moving with velocities u_1 and u_2 in the same direction as shown in figure. The ball deform and reform their shapes at the time of collision. At an instant when there will be maximum deformation, both the balls will have same velocity V

Let the balls separate with velocities v_1 and v_2 after collision

As per Newton's law of collision, coefficient of restitution (e) can be written as follows:-

$$e = \frac{J_r}{J_d} = \frac{v_{sep}}{v_{app}}$$

Here

$$v_{sep} = v_2 - v_1$$

$$v_{app} = u_1 - u_2$$

Thus $e = \dfrac{v_2 - v_1}{u_1 - u_2}$

Classification of collision.

On the Basis of orientation of collision

1) Head on collision.

In head-on collisions, the velocity of particles are along the same direction or opposite direction before and after the collision.

2) Oblique collision.

In oblique collision the velocity of particles are along different lines before and after the collision.

On the basis of kinetic energy conservation

1 Elastic collision :—

In elastic collision, particles regain their shape completely after the collision. This is an ideal condition that may not be possible in the real world. As the shape of particles before and after collisions is the same, the value of the coefficient of restitution is 1 for elastic collision. This

$$J_{ref} = J_{def}$$

And

$$v_{sep} = v_{app}$$

In elastic collision as $v_{sep} = v_{app}$, the kinetic energy of the system before and after the collision is the same.

2 Inelastic collision

In inelastic collision, the body will deform and will not be able to regain its shape completely.

As the shape of particle before and after the collision is not the same, the value of the coefficient of restitution is between 0 to 1 for inelastic collision.

During inelastic collisions, there is some loss of kinetic energy.

Thus the final kinetic energy of the system will be less than the initial kinetic energy.

Mathematically.

$J_{ref} < J_{def}$

And

$V_{sp} < V_{app}$.

3. Perfectly inelastic collisions.

In perfectly inelastic collisions, the velocity of seperation along the line of the collision is 0

Thus the coefficient of restitution is also zero in perfectly inelastic collisions and the loss of kinetic energy is maximum

The bodies will never seperate in perfectly inelastic collisions.

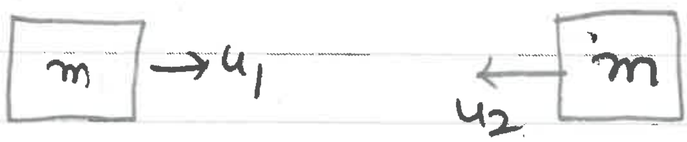
Mathematically

$J_{ref} = 0$
And
$V_{sep} = 0$

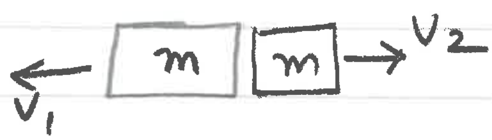
** If there is friction on the surface of the colliding bodies, kinetic energy is always lost during the collision, as friction is an impulsive force

Special cases of collisions.

Case 1



Before collision.



Let two particles with identical masses collide head-on
Let the collision be elastic.
Thus,

$e = 1$ and $m_1 = m_2 = m$

Let their velocities by u_1 and u_2 before collision and v_1 and v_2 after collision as shown in the figure

Consider the rightwards direction as positive

By using the principle of conservation of momentum we get the following

$$mu_1 - mu_2 = -mv_1 + mv_2$$

$$v_2 - v_1 = u_1 - u_2 \quad \text{---(i)}$$

Coefficient of restitution is given as follows.

$$e = \frac{V_{sep}}{V_{app}} = \frac{v_2 - (-v_1)}{u_2 - (-u_2)} = 1.$$

$$\frac{V_2 + V_1}{u_1 + u_2} = 1$$

$$\Rightarrow V_2 + V_1 = u_1 + u_2 \quad \text{---(ii)}$$

Adding equation (i) and (ii) we get the following:

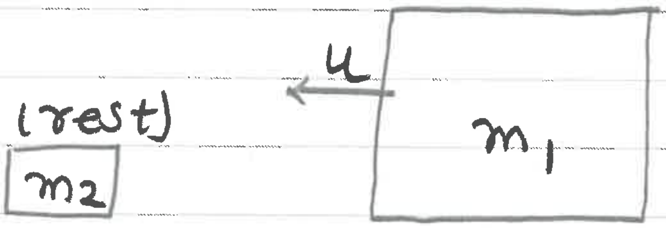
$$2V_2 = 2u_1$$
$$\Rightarrow V_2 = u_1$$

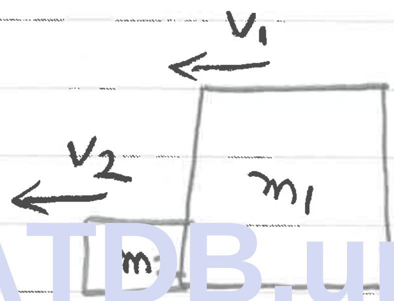
Substituting in equation (i), we get the following.

$$v_1 = u_2.$$

Thus the particle with the same masses exchange their velocity during a head-on unrestricted elastic collision

Case-2.





Let mass m_1 be very heavy as compared to mass m_2 $(m_1 >> m_2)$

This means,

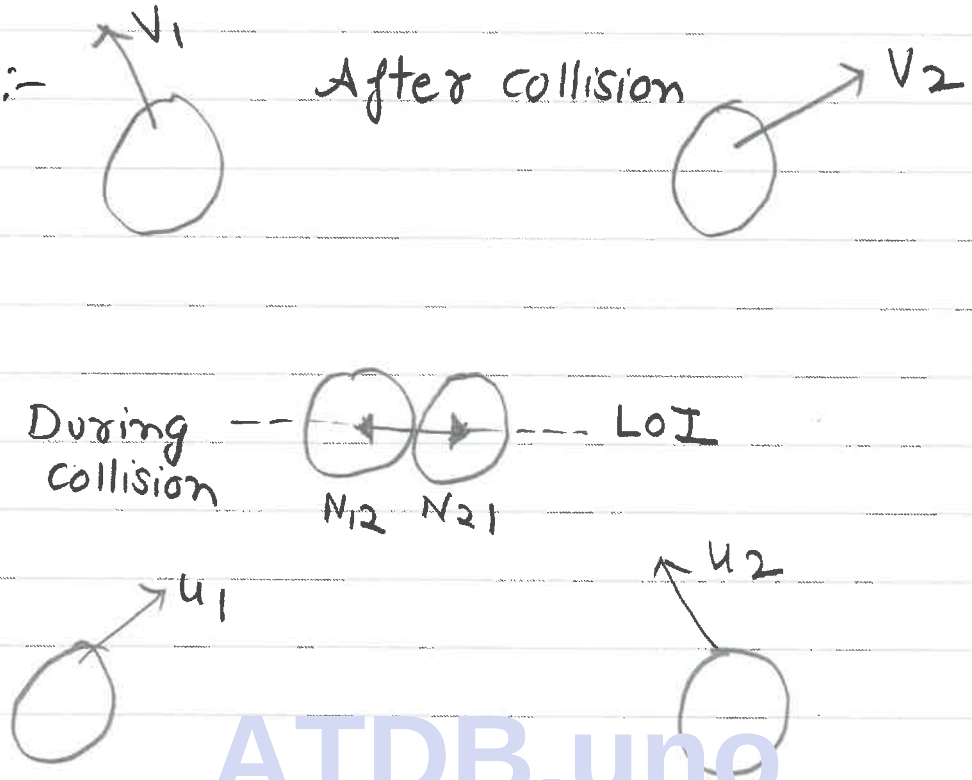
$$m_1 + m_2 \approx m_1 \quad \text{or} \quad \frac{m_2}{m_1} \to 0$$

Consider m_2 at rest initially and m_1 moving in the left direction with velocity u. Take the left direction as positive.

Let v_1 and v_2 be final velocities of masses m_1 and m_2 respectively.

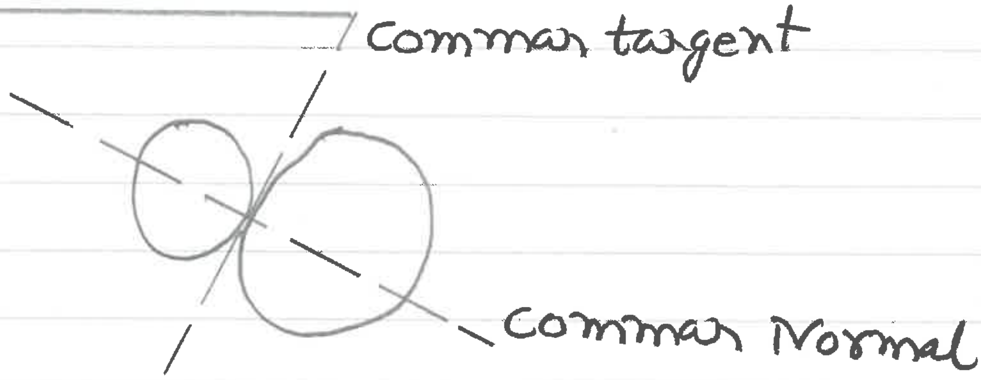
- The maximum momentum trasfers along the LOI during a collision.

Example:-



Let two balls with velocity u_1 and u_2 collide as shown in the figure. Even though their initial and final velocities are directed along different directions. the line of impact for this collision is along horizontal.

Oblique collision :-



By using the principle of conservation of momentum

$$m_1 u + 0 = m_1 v_1 + m_2 v_2$$

As m_2 is very small

$$u = v_1$$

This shows that, when one body is infinitely heavy with respect to the other, it retains it velocity past collision in a head-on collision.

Line of Impact

- When two bodies collide, the common normal to their surface at the point of contact is known as the line of impact.
- The line of impact for a circular/spherical body always passes through its centre point.
- The maximum impact during the collision acts along this line on both the bodies.
- Irrespective of the orientation of the colliding objects, the line of impact (LOI) always acts along the common normal.

- A pair of equal and opposite impulses acts along the comman normal direction.

- The momentum of the individual particles Changes along the comman normal direction

- No component of impulse acts along the comman tagent direction. The linear momentum of the individual particles remain unchanged along this direction.

- The net impulse ~~of~~ on System is zero during the collisions.

- The net momentum of both the particles remain conserved before ~~th~~ and after the collision in any direction

- The Coefficient of restitution can be applied along the line of impact.