SURE SHOT QUESTIONS

ARVIND ACADEMY

Chapter – 01

Relations and Functions

➢ MCQ

1. Soln. (c): (i) We know that every triangle is congruent to itself.

   $\therefore$ $(T_1, T_1) \in R$ for all $T_1 \in T$. Thus, R is reflexive.

   (ii) Let $(T_1, T_2) \in R \Rightarrow T_1$ is congruent to T_2.

   $\Rightarrow$ T_2 is congruent to T_1. $\therefore (T_2, T_1) \in R$

   Thus, R is symmetric.

   (iii) Let $(T_1, T_2) \in R$ and $(T_2, T_3) \in R$

   $\Rightarrow$ T_1 is congruent to T_2 and T_2 is congruent to T_3.

   $\therefore$ T_1 is congruent to $T_3 \Rightarrow (T_1, T_3) \in R$

   Thus, R is transitive.

   $\therefore$ R is an equivalence relation.

2. Soln. (b): Given $aRb \Rightarrow a$ is brother of b.

But $b \not\Re a$ [$\because$ b may or may not be brother of a]

$\therefore$ R is not symmetric.

Let aRb and bRc

$\Rightarrow$ a is brother of b and b is brother of c.

$\therefore$ a is brother of $c \Rightarrow (a, c) \in R$. $\therefore$ R is transitive.

3. Soln. (d): The smallest equivalence relation is the identity relation $R_1 = \{(1,1),(2,2),(3,3)\}$

   Then, two ordered pairs of two distinct elements can be added to give three more equivalence relations.

   $R_2 = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$

   Similarly R_3 and R_4.

   Finally the largest equivalence relation, that is the universal relation.

   $R_5 = \{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$

4. Soln. (b)

5. Soln. (b): Given aRb, $a \geq b$

   (i) Now $a \geq a$ is true for all real no.

   $\therefore$ R is reflexive.

   (ii) Let $(a, b) \in R$, $a \geq b$

Now $a \geq b$ but does not imply $b \geq a$.

$\therefore$ $(b, a) \notin R$ $\therefore$ R is not symmetric.

   (iii) Let $(a, b) \in R$ and

   $(b, c) \in R \Rightarrow a \geq b$ and $b \geq c$

$\therefore$ $a \geq c$ $\Rightarrow$ $(a, c) \in R$ $\therefore$ R is transitive.

6. Soln. (a): (1, 1), (2, 2), (3, 3) $\in$ R

$\therefore$ R is reflexive but it is not symmetric.

Also, R is transitive.

7. Soln. (c): $a * b = \frac{ab}{2}$

Let e be an identity element of * on Q – {0}.

$\therefore$ $a * e = a \,\forall\, a \in Q - \{0\} \Rightarrow \frac{ae}{2} = a$

$\Rightarrow$ $e = 2$

8. Soln. (c): As A contains 5 elements.

$\therefore$ For any one – one onto mapping $f : A \to B$, f(A) also contains 5 elements but B contains 6 elements.

$\therefore$ $f(A) \neq B$.

So, no one – one mapping from A to B can be onto.

9. Soln. (b): If $f : A \to B$ is a function, then $f(1)$ can be chosen in two ways, f(2) can be chosen in two ways,....., f(n) can be chosen in two ways.

Hence, f can be chosen in $2 \times 2 \times .... \times 2 = 2^n$ ways

In total there are 2^n functions possible. Out of these two function f_1 and f_2, defined as

$f_1(i) = a \ \forall \ i = 1, 2, ..., n$ and

$f_2(i) = b \ \forall \ i = 1, 2, ..., n$ are not surjective as range of f_1 is $\{a\} \neq B$ and f_2 is $\{b\} \neq B$.

Hence, the number of surjections from A to B is $2^n - 2$

10. Soln. (d): Since, $\frac{1}{x}$ is not defined for x = 0

$\therefore \quad f : R \to R$ can not be defined.

11. Soln. (a): $gof(x) = g(f(x)) = g(3x^2 - 5)$

$$= \frac{3x^2 - 5}{(3x^2 - 5)^2 + 1} = \frac{3x^2 - 5}{9x^4 - 30x^2 + 26}$$

12. Soln. (b): $f(x) = x^3$ cannot be onto as range of

$f = \{...., -27, -8, -1, 0, 1, , 8, 27, ...\} \neq Z$

$f(x) = 2x + 1$ is also not onto as

$R_f = \{...., -3, -1, 1, 3, ...\} \neq Z$

$f(x) = x^2 + 1$ is not one – one as

$f(x) = f(-x) = x^2 + 1$

And $f(x) = x + 2$ is one – one as

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ and it is onto also

$[\because \quad R_f = Z]$

Hence, $f(x) = (x + 2)$ is bijective.

13. Soln. (b): Let $y = f(x) = x^3 + 5 \Rightarrow x^3 = y - 5$

$\Rightarrow x = (y - 5)^{1/3} \quad \therefore \quad f^{-1}(x) = (x - 5)^{1/3}$

14. Soln. (a): Let $f(x) = y$ and $g(y) = z$

Then, $gof(x) = g(f(x)) = g(y) = z \Rightarrow (gof)^{-1}(z) = x$

Now, $f(x) = y$, $g(y) = z \Rightarrow f^{-1}(y) = x$ and $g^{-1}(z) = y$

$$(f^{-1} o g^{-1}) z = f^{-1}(g^{-1}(z)) = f^{-1}(y) = x$$

$\therefore \quad (gof)^{-1} = (f^{-1} o g^{-1})$

15. $f^{-1}(x) = \frac{1}{19} f(x)$

Soln. (a): Let $f(x) = y = \frac{3x + 2}{5x - 3} \Rightarrow x = \frac{2 + 3y}{5y - 3}$

$$\therefore \qquad f^{-1}(x) = \frac{2 + 3x}{5x - 3} = f(x)$$

16. Soln.

(b): Total number of reflexive relations on a set having n number of elements = $2^{n^2 - n}$

Here, n = 2

$\therefore$ Required number of reflexive relation = $2^{2^2 - 2}$

$= 2^{4-2} = 2^2 = 4$

17. Soln.

(b): Given, $R = R = \{(a, b) : a = b - 2, b > 6\}$

Since, $b > 6$, so $(2, 4) \notin R$

Also, $(3, 8) \notin R$ as $3 \neq 8 - 2$

And $(8, 7) \notin R$ as $8 \neq 7 - 2$

Now, for (6, 8), we have

8 > 6 and 6 = 8 – 2, which is true

$\therefore \ (6, 8) \in R$

18. Soln.

(c): Consider,

$R = [(x, y) : xy$ is the square number, $x, y \in N]$

As, xx = x^2, which is the square of natural number x.

$\Rightarrow \ (x, x) \in R$. So, R is reflexive.

19. Soln.

(c): Equivalence relations in the set {1, 2, 3} containing the elements (1, 2) and (2, 1) are

$R_1 = \{(1,2),(2,1),(1,1),(2,2),(3,3)\}$

$R_2 = \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2),(1,1),(2,2),(3,3)\}$

$\therefore$ Number of equivalence relations is 2.

20. Soln. (d): Given, $aRb,\ a,b \in Z$

Reflexive: For $a \in Z$, we have

$a^2 - 7a.a + 6a^2 = a^2 - 7a^2 + 6a^2$

$= 0 \Rightarrow (a,a) \in \mathrm{R}$

$\therefore$ Relation is reflexive

Symmetric : Since, $(6,1) \in \mathrm{R}$

As, $6^2 - 7\times6\times1 + 6\times1^2 = 36 - 42 + 6 = 0$

But $(1,6) \notin \mathrm{R}$.

$\therefore$ Relation is not symmetric.

21. Soln. (b): Equivalence relations in the set containing the element (1, 3) are

$R_1 = \{(1,1),(3,3),(1,3),(3,1),(5,5)\}$

$R_2 = \{(1,1),(3,3),(5,5),(1,5),(5,1),(3,5),(5,3),(1,3),(3,1)\}$

$\therefore$ There are 2 possible equivalence relations.

22. Soln. (c) : Given R = {(1, 2), (2, 1), (1, 1)} is a relation on set {1, 2, 3}

Reflexive: Clearly (2, 2), (3, 3) $\notin R$

$\therefore$ R is not a reflexive relation.

Symmetric : Now $(1,2) \in \mathrm{R}$ and $(2,1) \notin \mathrm{R}$ $\therefore$ R is symmetric

Transitive: Now,

$(2,1) \in \mathrm{R}$ and $(1,2) \in \mathrm{R}$ but $(2,2) \notin \mathrm{R}$

$\therefore$ R is not transitive relation.

R is symmetric, but neither reflexive nor transitive.

➢ Assertion-Reasoning (1 mark)

23. **Ans. (c)** A is true but R is false.

Explanation: $f(x)$ is a one-one function if

$f(x_1) = f(x_2)\, x_1 = x_2$

Hence R is false.

Let f(x_1) = $f(x_2)$ for some $x_1 = x_2 \in R$

$\Rightarrow (x_1)^3 = (x_2)^3$

$\Rightarrow x_1 = x_2$

Hence $f(x)$ is one-one.

24. Sol. **(c)** A is true but R is false.

Explanation: Assertion is true because for each element a ∈ A , |a - a| = 0 < 3, so (1, 1) ∈ R,(2,2) , ∈ R (3, 3) , ∈ R(4,4) therefore R is reflexive.

Reason is false because a relation R on the set A is said to be transitive if for (a, b) ∈R and (b, c) ∈R, we have (a, c) ∈ R

25. Sol. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation: Assertion Here, R = {(x, y) : y is divisible by x} is a relation in the set A = {1, 2, 3, 4, 5, 6}.For reflexive, we know that x is divisible by x, which is true for all $x \in A$.

∴ $(x, x) \in R$ for all $x \in A$.

So, R is reflexive. For symmetry, we observe that 6 is divisible by 2 i.e. (2, 6) ∈ R but 2 is not divisible by 6 i.e. (6, 2) ∈ R.

So, R is not symmetric.

For transitivity, let (x, y) ∈ R and (y, z) ∈ R

⇒ y is divisible by x and z is divisible by y.

⇒ z is divisible by x.

⇒ (x, z) ∈R

e.g. 2 is divisible by 1 and 4 is divisible by 2.

So, 4 is divisible by 1. So, R is transitive. Hence, R is not an equivalence relation.

26. Sol. **(d)** A is false but R is true.

Explanation: The assertion is false because relation R is not symmetric, (1, 2)∈ R but (2, 1) ∈R

The reason is true because for a relationship to be equivalence it must be reflexive, symmetric, and transitive.

27. Sol. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation: By definition, a Relation in R us to be refelexive if X R X, x Z

So R is true.

$$a - a = 0 \Rightarrow 2\, divides\, a - a \Rightarrow a\, R\, a$$

Hence, R is reflexive and A is true.

28. Sol. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation: For one to one function, if f(x) = f(y)then x = y

$\therefore 1 + x_1^2 = 1 + x_2^2$

$x_1^2 = x_2^2$

here, every element in the range maps to only one element in domain.

∴ $f(x)$ is strictly monoatomic function and one to one function.

29. Sol. **(d)** A is false but R is true.

Explanation: Assertion is false because distinct elements in N has equal images. for example

$f(1) = \frac{(1+1)}{2} = 1$

$f(2) = \frac{2}{2} = 1$

Reason is true because for injective function if elements are not equal then their images should be unequal.

30. Sol. **(d)** A is false but R is true.

Explanation: R = {(1, 3), (4, 2) (2, 7) (2, 3) (3, 1)}As (2, 3) R but (3, 2) R

So, set A is not symmetric.

31. Sol. **(d)** A is false but R is true.

Explanation: Assertion is false because every function is not invertible. The function which is one-one and onto i.e. bijective functions are invertible so reason is true.

32. Sol. **(d)** A is false but R is true.

Explanation: Assertion is false because f(1.9) = [1.9] = 1 and f(1.8) = [1.8] = 1. So distinct elements of domain have same image.

Therefore Greatest Integer function f: $R \to R$, given by f(x) = [x] is not one-one. The reason is true because by definition a function f: A B is said to be injective if distinct elements of domain has distinct images i.e. $f(a) = f(b) \Rightarrow a = b$.

➢ Case Study Question

33. **Sol.** (i) **(d)** Equivalence

Explanation: Equivalence

(ii) **(c)** R is Symmetric but neither reflexive nor transitive.
Explanation: R is Symmetric but neither reflexive nor transitive.
(iii) **(c)** Bijective
Explanation: Bijective
(iv) **(b)** R
Explanation: R
(v) **(c)** 2x - 2y + 5 = 0
Explanation: 2x - 2y + 5 = 0

34. Sol. (i) (c) Reflexive and transitive but not symmetric
Explanation: Reflexive and transitive but not symmetric
(ii) (b) 6^2
Explanation: 6^2
(iii) (c) None of these
Explanation: None of these
(iv) (a) 2^{12}
Explanation: 2^{12}
(v) (c) Reflexive and Transitive
Explanation: Reflexive and Transitive

➢ Questions

35. Sol. We have, $A=\{1,2,3,4,5,6\}$ and a relation R on A defined as $R=\{(a,b): b=a+1\}$

Reflexive: Let $(a,a)\in R$

$\Rightarrow a=a+1 \Rightarrow a-a=1 \Rightarrow 0=1$, which is not possible.

$\therefore$ $(a,a)\notin R \Rightarrow$ R is not reflexive.

Symmetric: Let $(a,b)\in R \Rightarrow b=a+1$ ....(i)

Now, if $(b,a)\in R$

$\Rightarrow a=b+1 \Rightarrow b=b+1+1$ (using (i))

$\Rightarrow b=b+2 \Rightarrow b-b=2 \Rightarrow 0=2,$

which is not possible

$\Rightarrow (b,a)\notin R \Rightarrow R$ *is not symmetric.*

Transitive: Let $(a,b)\in R$ and $(b,c)\in R$

$\Rightarrow b=a+1$ *and* $c=b+1 \Rightarrow c=a+1+1$

$\Rightarrow c=a+2\neq a+1 \Rightarrow (a,c)\notin R \Rightarrow R$ *is not transitive.*

36. Soln. We have A = $\{x\in Z: 0\le x\le 12\}$

$\therefore$ $A=\{0,1,2,3,.....,12\}$

Also,

$S=\{(a,b): a,b\in Z, |a-b| \text{ is divisible by } 3\}$

(i) Reflexive : For any $a\in A$.

$|a-a|=0$, which is divisible by 3

Thus, $(a,a)\in S$ $\therefore$ S is reflexive.

(ii) Symmetric: Let $(a,b)\in S$

$\Rightarrow$ $|a-b|$ is divisible by 3.

$\Rightarrow$ $|b-a|$ is divisible by 3 $\Rightarrow (b,a)\in S$ i.e., $(a,b)\in S \Rightarrow (b,a)\in S$

$\therefore$ S is symmetric.

(iii) Transitive:

Let $(a,b)\in S$ and $(b,c)\in S$

$\Rightarrow$ $|a-b|$ is divisible by 3 and $|b-c|$ is divisible by 3.

$\Rightarrow$
$(a-b)=\pm 3k_1$ *and* $(b-c)=\pm 3k_2; \forall k_1, k_2\in N$

$\Rightarrow$ $(a-b)+(b-c)=\pm 3(k_1+k_2)$

$\Rightarrow$ $(a-c)=\pm 3(k_1+k_2); \forall k_1, k_2\in N$

$\Rightarrow$ $|a-c|$ is divisible by 3 $\Rightarrow (a,c)\in S$ $\therefore$ S is Transitive.

Hence, S is an equivalence relation.

37. Soln. f(x) is not one-one (i.e., injective)

$$As\ f(-x)=\frac{(-x)^2}{1+(-x)^2}=\frac{x^2}{1+x^2}=f(x)$$

But $x\neq -x$

For the function to be an onto function, let $y=f(x)$

$$\Rightarrow \quad \frac{x^2}{1+x^2}=y$$

$$\Rightarrow \quad x^2=\frac{y}{1-y}$$

$$\Rightarrow \quad \frac{y}{1-y}\ge 0,\ y\neq 1$$

$$\Rightarrow \quad 0\le y\le 1$$

$\Rightarrow$ Range of f(x) is not equal to its codomain.
Hence, it is not onto.
That is, the given function is neither one – one nor an onto function.

38. Soln. R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (3, 1)}

Clearly R is reflexive.

R is not symmetric since

$(1,2) \in R\,but\,(2,1) \notin R$

$(2,3) \in R\,but\,(3,2) \notin R$

$(3,1) \in R\,but\,(1,3) \notin R$

Also, R is not transitive.

Since $(1,2) \in R, (2,3) \in R\,but\,(1,3) \notin R$.

39. Soln. f is not injective, since

$f(x) = f(y)$

$\Rightarrow \quad x^2 + x = y^2 + y$

$\Rightarrow \quad x^2 + x + \frac{1}{4} = y^2 + y + \frac{1}{4}$

$\Rightarrow \quad \left(x + \frac{1}{2}\right)^2 = \pm\left(y + \frac{1}{2}\right)^2$

$\Rightarrow \quad x + \frac{1}{2} = \pm\left(y + \frac{1}{2}\right)$

$\Rightarrow \quad x + \frac{1}{2} = y + \frac{1}{2}$

or $\quad x + \frac{1}{2} = -y - \frac{1}{2}$

Thus, f(x) = f(y) does not give the unique solution x = y but also gives x = -y – 1.

Thus, f is not injective.

f is not surjective, since clearly $f(x) = x^2 + x \geq 0$ for all $x \in Z$.

Thus, negative integers do not have pre – images in Z.

Therefore, f is not an onto function.

40. Sol. The given function is $f : R \to R$ such that

$f(x) = 4x^3 + 7$

One – one: Given that

$f(x) = 4x^3 + 7$

Let $\quad f(x_1) = f(x_2), \forall x_1, x_2 \in R$

$\Rightarrow \quad 4x_1^3 + 7 = 4x_2^3 + 7$

$\Rightarrow \quad 4x_1^3 = 4x_2^3$

$\Rightarrow \quad x_1^3 - x_2^3 = 0$

$\Rightarrow \quad (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$

$\Rightarrow$ *Either* $\quad x_1 - x_2 = 0 \quad$ .....(i)

or $\quad x_1^2 + x_1x_2 + x_2^2 = 0 \quad$ ....(ii)

But equation (ii) is not possible as $x_1, x_2 \in R$

$\therefore \quad x_1 - x_2 = 0$

$\Rightarrow \quad x_1 = x_2$

$\therefore$ We have shown that

$f(x_1) = f(x_2)$

$\Rightarrow \quad x_1 = x_2, \forall\, x_1, x_2 \in R$

$\therefore\ f(x)$ is a one – one function.

41. Soln. $f : R \to R$ is given by

$f(x) = x^3 + 1$

For injective:

Let $\quad$ f(x) = f(y)

$\Rightarrow \quad x^3 + 1 = y^3 + 1$

$\Rightarrow \quad x^3 = y^3$

$\Rightarrow \quad x = y$

Thus, $f(x) = f(y)$

$\Rightarrow \quad x = y\ \forall\ x, y \in R$

For surjective:

Let $\quad y \in R$

Then, $\quad f(x) = y$

$\Rightarrow \quad x^3 + 1 = y$

$\Rightarrow \quad x^3 = y - 1$

Thus, for any $y \in R$,

$x = (y-1)^{1/3}$ is real number and

$f(x) = f[(y-1)^{1/3}]$

$= \left[(y-1)^{1/3}\right]^3 + 1$

$= (y-1) + 1 = y$

$\Rightarrow \quad f$ is surjective.

Therefore, f is bijective.

42. Sol. Here, R = {(a, b) : b = a + 1}

$\therefore$ R = {(a, a + 1) : a, a + 1 $\in$ (1, 2, 3, 4, 5, 6)}

$\Rightarrow \quad R = \{(1,2)(2,3), (3,4), (4,5), (5,6)\}$

(i) R is not reflexive as $(a,a) \notin R \ \forall a$

(ii) R is not symmetric as $(1, 2) \in R$ but $(2, 1) \notin R$

(iii) R is not transitive as $(1,2) \in R, (2,3) \in R$ but $(1,3) \notin R$

43. Sol. Given, $R = \{(a,b) : a, b \in A, |a-b|$ is divisible by 4$\}$

Reflexivity: For any $a \in A$

$|a-a| = 0$, which is divisible by 4

$(a,a) \in R$

So, R is reflexive.

Symmetry: Let $(a,b) \in R$

$\Rightarrow \quad |a-b|$ is divisible by 4

$\Rightarrow \quad |b-a|$ is divisible by 4

$[\because |a-b| = |b-a|]$

$\Rightarrow \quad (b,a) \in R$

So, R is symmetric.

Transitive: Let $(a,b) \in R$ and $(b,c) \in R$

$\Rightarrow |a-b|$ is divisible by 4

$\Rightarrow |a-b| = 4k$

$\therefore \quad a-b = \pm 4k, \ k \in Z \quad$ ......(i)

Also, $|b-c|$ is divisible by 4

$\Rightarrow |b-c| = 4m$

$\therefore \quad b-c = \pm 4m, \ m \in Z \quad$ ......(ii)

Adding equations (i) and (ii)

$a-b+b-c = \pm 4(k+m)$

$\Rightarrow \quad a-c = \pm(k+m)$

$|a-c|$ is divisible by 4,

$\Rightarrow \quad (a,c) \in R$

So, R is transitive.

$\Rightarrow$ R is reflexive, symmetric and transitive.

$\therefore$ R is an equivalence relation.

Let x be an element of R such that $(x,1) \in R$

Then $|x-1|$ is divisible by 4

$x-1 = 0, 4, 8, 12, \ldots\ldots$

$\Rightarrow \quad x = 1, 5, 9 \quad (\because x \le 12)$

$\therefore$ Set of all elements of A which are related to 1 are {1, 5, 9}.

Equivalence class of 2 i.e.,

$[2] = \{(a,2) : a \in A, |a-2|$ is divisible by $4\}$

$\Rightarrow \quad |a-2| = 4k$ (k is whole number, $k \le 3$)

$\Rightarrow \quad a = 2, 6, 10$

44. Sol. Let x, $y \in R$ such that $f(x) = f(y)$

$\therefore \quad f(x) = f(y)$

If x and y are odd, then

$f(x) = f(y)$

$\Rightarrow \quad x+1 = y+1$

$\Rightarrow \quad x = y$

If x and y are even, then

$f(x) = f(y)$

$\Rightarrow \quad x-1 = y-1$

$\Rightarrow \quad x = y$

If x is odd and y is even, then

$f(x) = x+1$ is even and $f(y) = y+1$ is odd.

$\therefore \quad x \ne y \Rightarrow f(x) \ne f(y)$

Hence, $f : N \to N$ is one – one

Also, $\quad f(1) = 1+1 = 2$

$f(1) = 2 \quad (\because 1$ is odd)

If x is odd number, then $\exists$ an even natural number, $x+1 \in N$ such that,

$f(x+1) = x+1-1$

$= x$

If x is even number, then there exist a odd natural number $x-1 \in N$ such that,

$f(x-1) = x-1+1$

$= x$

Hence for every $y \in N \ \exists \ x \in N$ such that f(x) = y, so f is onto.

Hence f is both one – one and onto.

45. Sol.

$\frac{1}{2} \ne \left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}, \frac{1}{2}\right) \ne R$.

Hence, R is not reflexive.

46. Sol. Let $(a,b) \in N \times N$

Then,

$\because \quad a^2 + b^2 = a^2 + b^2$

$\therefore \quad (a,b) \ R \ (a,b)$

Hence R is reflexive.

Let $(a,b), (c,d) \in N \times N$ be such that

$(a,b) \ R \ (c,d)$

$\Rightarrow \quad a^2 + d^2 = b^2 + c^2$

$\Rightarrow \quad c^2 + b^2 = d^2 + a^2$

$\Rightarrow \quad (c,d) \ R \ (a,b)$

Hence, R is symmetric.

Let $(a,b),(c,d),(e,f)\in N\times N$ be such that

$(a,b)R(c,d),(c,d)R(e,f)$.

$\Rightarrow \quad a^2+d^2=b^2+c^2 \quad .....(i)$

and $\quad c^2+f^2=d^2+e^2 \quad .....(ii)$

Adding eqn. (i) and (ii),

$\Rightarrow a^2+d^2+c^2+f^2=b^2+c^2+d^2+e^2$

$\Rightarrow \quad a^2+f^2 = b^2+e^2$

$\Rightarrow \quad (a,b)R(e,f)$

Hence, R is transitive

Since, R is reflexive, symmetric and transitive.

Therefore, R is an equivalence relation.

47. Sol. Given a relation S in N x N, defined as $(a,b)S(c,d)$, if $a+d=b+c$.

Reflexive Let (a, b) be any arbitrary element of N x N

i.e., $(a,b)\in N\times N$, where $a,b\in N$

Now, as $a+b=b+a$

[$\because$ addition is commutative]

$\therefore \quad (a,b)S(a,b)$

So, S is reflexive.

Symmetric Let $(a,b),(c,d)\in N\times N$, such that (a, b) S(c, d). Then, a + d = b + c

$\Rightarrow \quad b+c=a+d\Rightarrow c+b=d+a$

$\Rightarrow \quad (c,d)S(a,b)$

So, S is symmetric.

Transitive Let (a, b), (c, d), (e, f) $\in N\times N$ such that

$(a,b)S(c,d)$ and $(c,d)S(e,f)$.

Then, $a+d=b+c$ and $c+f=d+e$

On adding the above equations, we get

$a+d+c+f=b+c+d+e$

$\Rightarrow \quad a+f=b+e\Rightarrow(a,b)S(e,f)$

So, S is transitive.

Thus, S is reflexive, symmetric and transitive.

Hence, S is an equivalence relation.

48. Sol. Given $(a,b)R(c,d)$ as $ad(b+c)=bc(a+d)$

$\therefore \quad \forall\, a,b\in N$,

Or $\quad ab(b+a)=ba(a+b)$

Or $\quad (a,b)R(a,b)$

$\therefore \quad R$ is reflexive $\quad$ ....(i)

Let $(a,b)R(c,d)$ for $(a,b),(c,d)\in N\times N$

$\therefore \quad ad(b+c)=bc(a+d) \quad$ ......(ii)

Also, $\quad (c,d)R(a,b)$

$\because\ cb(d+a)=da(c+b)$ [By commutation of addition and multiplication on N]

$\therefore \quad$ R is symmetric. $\quad$ ....(iii)

Let (a, b) R (c, d) and (c, d) R (e, f) for a, b, c, d, e, f $\in$ N

$\therefore \quad ad(b+c)=bc(a+d) \quad$ ......(iv)

and $\quad cf(d+e)=de(c+f) \quad$ ......(v)

Dividing eqn. (iv) by abcd and eqn. (v) by cdef

i.e., $\quad \frac{1}{c}+\frac{1}{b}=\frac{1}{d}+\frac{1}{a}$

and $\quad \frac{1}{e}+\frac{1}{d}=\frac{1}{f}+\frac{1}{c}$

On adding, we get

$$\frac{1}{c}+\frac{1}{b}+\frac{1}{e}+\frac{1}{d}=\frac{1}{d}+\frac{1}{a}+\frac{1}{f}+\frac{1}{c}$$

Or $\quad af(b+e)=be(a+f)$

Hence

$(a,b)R(e,f)$

$\therefore \quad$ R is transitive.

From equation (i), (iii) and (iv), R is an equivalence relation.

49. Sol. The given relation is R = {(a, b) : |a – b| is even} defined on set A = {1,2,3,4,5}.

Reflexive As |x – x| = 0 is even, $\forall\ x\in A$

Therefore, R is reflexive.

Symmetric Let $\{x,y\}\in R \Rightarrow |x-y|$ is even

[by the definition of given relation]

$\Rightarrow |y-x|$ is also even $[\because |a|=|-a|, \forall a\in R]$

$\Rightarrow \ (y,x)\in R$

Thus, $(x,y)\in R$

$\Rightarrow \ (y,x)\in R,\ \forall\ x,y\in A$

Therefore, R is symmetric.

Transitive Let $(x,y)\in R$ and (y, z) $\in$ R

$\Rightarrow$ |x – y| is even and |y – z| is even.

[by using definition of given relation]

Now, |x – y| is even.

$\Rightarrow$ x and y both are even or odd.

And |y – x| is even.

$\Rightarrow$ y and z both are even or odd.

Clearly, two cases arises.

Case I When y is even. Then, both x and z are even.

$\Rightarrow$ $|x-z|$ is even.

[$\because$ difference of two even numbers is even]

$\Rightarrow$ $(x,z)\in R$

Case II When y is odd.

Then, both x and z are odd.

$\Rightarrow$ $|x-z|$ is even.

[$\because$ difference of two odd numbers is even]

$\Rightarrow$ $(x,z)\in R$

Thus, $(x,y)\in R$ and $(y,z)\in R$

$\Rightarrow$ $(x,z)\in R,\ \forall\ x,y,z\in A$

Therefore, R is transitive.

Since, R is reflexive, symmetric and transitive, so it is an equivalence relation.

50. Soln. $f(x_1)=f(x_2)$

Or $\left(\dfrac{x_1-2}{x_1-3}\right)=\left(\dfrac{x_2-2}{x_2-3}\right)$

Or $x_1=x_2$

$\therefore$ f is a one – one function.

Let, $y=f(x)=\dfrac{x-2}{x-3}$

Or $x=\dfrac{3y-2}{y-1}$, where $y\neq 1$ and $x\neq 3$

$\therefore$ For each $y\in B$ there exists $x\in A$ such that f(x) = y or f is onto

f is a one – one and onto function.

Or f is bijective function.

$f^{-1}:B\to A\ with\ f^{-1}(x)=\dfrac{3x-2}{x-1}$

51. Soln.

Given $f(x)=\dfrac{4x+3}{6x-4}$

To show f is one – one:

Let, $f(x_1)=f(x_2)$,

Then $\dfrac{4x_1+3}{6x_1-4}=\dfrac{4x_2+3}{6x_2-4}$

Or

$(4x_1+3)(6x_2-4)=(6x_1-4)(4x_2+3)$

Or

$24x_1x_2-16x_1+18x_2-12=24x_1x_2+18x_1-16x_2-12$

Or $-16x_1+18x_2=18x_1-16x_2$

Or $-16x_1-18x_1=-18x_2-16x_2$

Or $-34x_1=-34x_2$

Or $x_1=x_2$

Or f is one – one.

To show f is onto:

Let, $y\in B$

$\therefore$ $y=f(x)$

Or $y=\dfrac{4x+3}{6x-4}$

Or $y(6x-4)=4x+3$

Or $6xy-4y=4x+3$

Or $6xy-4x=4y+3$

Or $x(6y-4)=4y+3$

Or $x=\dfrac{4y+3}{6y-4}\in B=R-\left\{\dfrac{2}{3}\right\}$

Or For every value of y except $y=\left\{\dfrac{2}{3}\right\}$, there is a pre image $x=\dfrac{4y+3}{6y-4}=g(y)$.

Or $x\in A$

Or f is onto.

To find f^{-1}:

Since, f is one – one and onto, therefore f is invertible.

Thus, $f^{-1}(x)=\dfrac{4x+3}{6x-4}=g(x)$

52. Soln.

(i) $f(x)=x+|x|$

$=\begin{cases}x+x, & if\ \ x\geq 0\\ x-x, & if\ \ x<0\end{cases}=\begin{cases}2x, & if\ \ x\geq 0\\ 0, & if\ \ x<0\end{cases}$

Thus,

$f(x)=2x\geq 0\ \forall\ x\geq 0$ and $f(x)=0\ \forall\ x<0$

$\therefore$ f(x) can't be negative for any $x\in R$

Thus, f is not onto.

(ii) We have $f(x)=x+1\ \forall\ x\in R$

For any $y\in R, y=f(x)\Rightarrow y=x+1\Rightarrow x=y-1$

$\therefore$ $f(y-1)=y-1+1=y$

Hence, f is onto.

53. Soln.

$R=\{(a,b):a^2+b^2=25\}$ be a relation on Z.

To domain of R is the value of $a\in Z$ that satisfy the relation $a^2+b^2=25$

$\Rightarrow\ a^2=25-b^2\Rightarrow a=\pm\sqrt{25-b^2}$

Domain of $R=\{0,\pm3,\pm4,\pm5\}$

54. Soln. If $f:A\rightarrow B$ is such that $y\in B$, then

$f^{-1}(y)=\{x\in A:f(x)=y\}$

In other words, $f^{-1}(y)$ is the set of pre – images of y.

Let $f^{-1}(17)=x$

$\Rightarrow\quad f(x)=17\Rightarrow x^2+1=17$

$\Rightarrow\quad x^2=17-1=16\Rightarrow x=\pm4$

$\therefore\quad f^{-1}(17)=\{-4,4\}$

Again, let $f^{-1}(-3)=x$, then

$f(x)=-3\Rightarrow x^2+1=-3$

$\Rightarrow\ x^2=-4\ \Rightarrow x=\sqrt{-4}$

Clearly no solution is available in R.

So, $f^{-1}(-3)=\phi$.

55. Soln.

Here, $f:R^+\rightarrow R^+$ defined by $f(x)=\dfrac{1}{2x}$

One – One: Let $x_1,x_2\in R^+$ (domain)

Now, $f(x_1)=f(x_2)\Rightarrow\dfrac{1}{2x_1}=\dfrac{1}{2x_2}$

$\Rightarrow\ 2x_1=2x_2\Rightarrow x_1=x_2$

$\therefore$ f is one – one.

Onto : Let $y\in R^+$ (co-domain) be any arbitrary element then $y\neq0$

Let $y=f(x)$

$\Rightarrow\quad y=\dfrac{1}{2x}\Rightarrow x=\dfrac{1}{2y}\in R^+$

$\therefore$ f is onto.

Hence, f is bijective where $\dfrac{1}{2y}$ is non zero real number.

Hence, each element of co-dominan (R^+) is the image of some element of domain (R^+).

56. Soln.

Here, $A=\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$ and $B=[-1,1]$

Also $f:A\rightarrow B$ such that f(x) = sinx

$\therefore$ f is one – one.

$\because\ f(x_1)=f(x_2)\Rightarrow\sin x_1=\sin x_2$

$\Rightarrow\ x_1=x_2\qquad\left[\because x_1,x_2\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]\right]$

Also, range (f) = [-1, 1] = B. So, f is onto.

Thus, f is one – one and onto and hence bijective.

57. Soln.

We have $f:R\rightarrow R$ defined by $f(x)=x^3+x$

Let $x_1,x_2\in R$ such that $f(x_1)=f(x_2)$

$\Rightarrow x_1^3+x_1=x_2^3+x_2\Rightarrow x_1^3-x_2^3+x_1-x_2=0$

$\Rightarrow(x_1-x_2)(x_1^2+x_1x_2+x_2^2+1)=0$

$\Rightarrow\ x_1-x_2=0$

$[\because x_1^2+x_1x_2+x_2^2\geq0$ for all $x_1,x_2\in R$

$\therefore x_1^2+x_1x_2+x_2^2+1\geq1$ *for all* $x_1,x_2\in R]$

$\Rightarrow\quad x_1=x_2$

$\Rightarrow\quad f$ *is one – one*

Let y be any arbitrary element of R, then there exists $x\in R$ such that f(x) = y.

$\Rightarrow\ x^3+x=y\Rightarrow x^3+x-y=0$

Since, odd degree equation has atleast one real root.

Thus, for every value of y, the equation $x^3+x-y=0$ has a real root α, such that

$\alpha^3+\alpha-y=0$

$\Rightarrow f(\alpha)=y$

Thus, for every $y\in R,\exists\ \alpha\in R$ such that $f(\alpha)=y$

So, f is onto function.

Hence, $f:R\rightarrow R$ is a bijection.

58. Soln.

Here, R = {(a, b) : a is a factor of b for a, b$\in N$

Reflexive: Let a be an arbitrary element of N then, clearly, a is a factor of a.

$\therefore (a,a) \in R\ \forall a \in N$ $\therefore$ R is reflexive.

Symmetric: Clearly 2 and 6 are natural numbers and 2 is a factor of 6.

$\therefore$ $(2,6) \in R$ but 6 is not a factor of $2 \Rightarrow (6,2) \notin R$

Thus, $(2,6) \in R$ but $(6,2) \notin R$ $\therefore$ R is not symmetric.

Transitive: Let $a, b, c \in N$

Now $(a,b) \in R$ and $(b,c) \in R$

$\Rightarrow$ (a is a factor of b) and (b is a factor of c)

$\Rightarrow$ b = ad and c = be for some d, $e \in N$

$\Rightarrow$ $c = (ad)e = a(de)$ [By associative law]

$\Rightarrow$ a is a factor of $c \Rightarrow (a,c) \in R$

Thus, $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$

$\therefore$ R is transitive.

59. Soln.

(i) Here, $f : R \to R$ given by f(x) = sinx

Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow$ $\sin x_1 = \sin x_2 \Rightarrow x_1 = n\pi + (-1)^n x_2 \Rightarrow x_1 = x_2$

$\therefore$ f is not one – one.

Let $y \in R$ be any arbitrary element, then there exists $x \in R$ such that f(x) = y

$\Rightarrow \sin x = y \Rightarrow x = \sin^{-1} y$

$\because$ $For\ y > 1, x \notin R$ (domain)

$\therefore$ f is not onto.

Hence, f is not a bijective function.

(ii) $f : R \to R$ defined by $f(x) = \sin^2 x + \cos^2 x$

Since, $f(x) = \sin^2 x + \cos^2 x = 1$

Now, f(x) = 1 is a constant function and we know that constant function is neither injective nor surjective.

$\therefore$ f is neither one – one nor onto.

60. Soln.

We have given, $A = \{1,2,3,4\}$

(i) Consider,

$R_1 = \{(1,1),(1,2),(2,3),(2,2),(1,3),(3,3)\}$

As, (1, 1), (2, 2), (3, 3) lie in R_1

$\therefore$ R_1 is reflexive.

Also, $(1,2) \in R_1, (2,3) \in R_1 \Rightarrow (1,3) \in R_1$

So, R_1 is also transitive.

Since, $(2,3) \in R_1$ but $(3,2) \notin R_1$.

So, it is not symmetric.

(ii) Consider, $R_2 = \{(1,2),(2,1)\}$

As, $(1,2) \in R_2$ and $(2,1) \in R_2$

So, it is symmetric but it is neither reflexive nor transitive.

(iii) Consider,

$R_3 = \{(1,2),(2,1),(1,1),(2,2),(3,3),(1,3),(3,1),(2,3),(3,2)\}$

Hence, R_3 is reflexive, symmetric and transitive.

61. Soln.

(i) Here, $x = \frac{1}{2}$, which is rational and satisfying first condition.

$\therefore$ $f\left(\frac{1}{2}\right) = 1$

(ii) Here, $x = \sqrt{2}$, which is irrational and satisfying second condition.

$\therefore$ $f(\sqrt{2}) = -1$

(iii) Here, $x = \pi$, which is irrational and satisfying second condition.

$\therefore$ $f(\pi) = -1$

(iv) Here, $x = 2 + \sqrt{3}$, which is irrational and satisfying second condition.

$\therefore$ $f(2+\sqrt{3}) = -1$

Clearly, f(x) is many one as f(x) = -1 for $x = \sqrt{2}$ and $2+\sqrt{3}$.

And f(x) takes values only 1 and -1.

12 MATERIAL 12TH

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SURE SHOT QUESTIONS

ARVIND ACADEMY

Chapter – 02 (Solution)

Inverse Trigonometric Functions

➢ MCQ (1 mark)

1. Soln. (c): Principal value branch of $\cos^{-1} x$ is $[0, \pi]$

2. Soln. (d): Principal value branch of $\operatorname{cosec}^{-1} x$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

3. Soln. (b): $3\tan^{-1} x + \cot^{-1} x = \pi$

$$\Rightarrow 2\tan^{-1} x + \tan^{-1} x + \cot^{-1} x = \pi \Rightarrow 2\tan^{-1} x + \frac{\pi}{2} = \pi$$

$$\Rightarrow 2\tan^{-1} x = \pi - \frac{\pi}{2} = \frac{\pi}{2} \Rightarrow \tan^{-1} x = \frac{\pi}{4} \Rightarrow x = 1$$

4. Soln. (d):

$$\sin^{-1}\left(\cos\left(\frac{33\pi}{5}\right)\right) = \sin^{-1}\left(\cos\left(6\pi + \frac{3\pi}{5}\right)\right)$$

$$= \sin^{-1}\left(\cos\frac{3\pi}{5}\right) = \sin^{-1}\left[\cos\left(\frac{\pi}{2} + \frac{\pi}{10}\right)\right] = \sin^{-1}\left(-\sin\frac{\pi}{10}\right)$$

$$= -\sin^{-1}\left(\sin\frac{\pi}{10}\right) = -\frac{\pi}{10}$$

5. Soln. (a): We know, $0 \le \cos^{-1}(2x-1) \le \pi$

$$\Rightarrow -1 \le 2x - 1 \le 1 \Rightarrow 0 \le 2x \le 2 \Rightarrow 0 \le x \le 1$$

$\therefore$ Domain of f(x) is [1, 2].

6. Soln. (a): We know, $\frac{-\pi}{2} \le \sin^{-1}\sqrt{x-1} \le \frac{\pi}{2}$

$$\Rightarrow -1 \le \sqrt{x-1} \le 1 \Rightarrow 0 \le x - 1 \le 1 \Rightarrow 1 \le x \le 2$$

$\therefore$ Domain of f(x) is [1, 2]

7. Soln. (b): $\cos\left(\sin^{-1}\frac{2}{5} + \cos^{-1} x\right) = 0$

$$\Rightarrow \sin^{-1}\frac{2}{5} + \cos^{-1} x = \cos^{-1} 0 = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\frac{2}{5} + \frac{\pi}{2} - \sin^{-1} x = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1} x = \sin^{-1}\left(\frac{2}{5}\right) \Rightarrow x = \frac{2}{5}$$

8. Soln. (c): Let $2\tan^{-1}(0.75) = \theta \Rightarrow 0.75 = \tan\left(\frac{\theta}{2}\right)$

$$\therefore \sin(2\tan^{-1}(0.75)) = \sin\theta = \frac{2\tan\theta/2}{1+\tan^{2}\theta/2}$$

$$= \frac{2 \times 0.75}{1+(0.75)^2} = \frac{1.50}{1.5625} = 0.96$$

9. Soln. [illegible]: $\cos^{-1}\left(\cos\frac{3\pi}{2}\right) = \cos^{-1}\left(\cos\left(\pi + \frac{\pi}{2}\right)\right)$

$$= \cos^{-1}\left(-\cos\frac{\pi}{2}\right) = \pi - \cos^{-1}\left(\cos\frac{\pi}{2}\right) \Rightarrow \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

10. Soln. (b):

$$2\sec^{-1}(2) + \sin^{-1}\left(\frac{1}{2}\right) = 2\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right)$$

$$= \cos^{-1}\left(\frac{1}{2}\right) + \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right)$$

$$= \cos^{-1}\left(\frac{1}{2}\right) + \frac{\pi}{2} = \frac{\pi}{3} + \frac{\pi}{2} = \frac{5\pi}{6}$$

11. Soln. (a): $\tan^{-1} x + \tan^{-1} y = \frac{4\pi}{5}$

$$\Rightarrow \frac{\pi}{2} - \cot^{-1} x + \frac{\pi}{2} - \cot^{-1} y = \frac{4\pi}{5}$$

$$\Rightarrow \pi - \frac{4\pi}{5} = \cot^{-1} x + \cot^{-1} y \Rightarrow \cot^{-1} x + \cot^{-1} y = \frac{\pi}{5}$$

12. Soln. (d):

$$\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$\Rightarrow 2\tan^{-1}a+2\tan^{-1}a=2\tan^{-1}x$$

$$\Rightarrow 4\tan^{-1}(a)=2\tan^{-1}x\Rightarrow 2\tan^{-1}a=\tan^{-1}x$$

$$\Rightarrow \tan^{-1}\left(\frac{2a}{1-a^2}\right)=\tan^{-1}x\Rightarrow x=\frac{2a}{1-a^2}$$

13. Soln. (d):

$$\cot\left(\cos^{-1}\left(\frac{7}{25}\right)\right)=\cot\left(\cot^{-1}\left(\frac{7}{24}\right)\right)=\frac{7}{24}$$

14. Son. (b): Let $\tan\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt5}\right)=\theta$

$$\Rightarrow \cos^{-1}\frac{2}{\sqrt5}=2\tan^{-1}\theta\Rightarrow \cos^{-1}\frac{2}{\sqrt5}=\cos^{-1}\left(\frac{1-\theta^2}{1+\theta^2}\right)$$

$$\Rightarrow \frac{2}{\sqrt5}=\frac{1-\theta^2}{1+\theta^2}\Rightarrow 2+2\theta^2=\sqrt5-\sqrt5\theta^2$$

$$\Rightarrow \theta^2(\sqrt5+2)=\sqrt5-2\Rightarrow \theta^2=\frac{\sqrt5-2}{\sqrt5+2}$$

$$\Rightarrow \theta^2=(\sqrt5-2)^2\Rightarrow \theta=\sqrt5-2$$

15. Soln. (a):

$$2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}=2\tan^{-1}x+2\tan^{-1}x=4\tan^{-1}x$$

16. Soln. (c):

$$\cos^{-1}\alpha+\cos^{-1}\beta+\cos^{-1}\gamma=3\pi \because 0\le\cos^{-1}x\le\pi$$

$$\Rightarrow \cos^{-1}\alpha=\cos^{-1}\beta=\cos^{-1}\gamma=\pi$$

$$\Rightarrow \alpha=\beta=\gamma=-1$$

$$\therefore \alpha(\beta+\gamma)+\beta(\gamma+\alpha)+\gamma(\alpha+\beta)$$

$$=-1(-1-1)+(-1)(-1-1)+(-1)(-1-1)$$

$$=2+2+2=6$$

17. Soln. (a): $\sqrt{1+\cos 2x}=\sqrt2\cos^{-1}(\cos x)$

$$\Rightarrow \sqrt2\,|\cos x|=\sqrt2\cos^{-1}(\cos x)\Rightarrow |\cos x|=x$$

$\therefore$ No solution.

18. Soln. (c):

$$\cos^{-1}x>\sin^{-1}x\Rightarrow \frac{\pi}{2}-\sin^{-1}x>\sin^{-1}x$$

$$\Rightarrow \frac{\pi}{2}>2\sin^{-1}x\Rightarrow \sin^{-1}x<\frac{\pi}{4}\Rightarrow x<\sin\frac{\pi}{4}\Rightarrow x<\frac{1}{\sqrt2}$$

19. Ans. (c): $\sec^{-1}x$ is defined if $x\le-1$ or $x\ge1$.

Hence, $\sec^{-1}2x$ will be defined if

$$x\le-\frac{1}{2}\ or\ x\ge\frac{1}{2}$$

The range of the function $\sec^{-1}x$ is $[0,\pi]-\left\{\frac{\pi}{2}\right\}$

Hence, A is true and R is false.

20. Ans. (d): We have,

$$\sin\left[\frac{\pi}{3}-\sin^{-1}\left(\frac{-1}{2}\right)\right]$$

$$=\sin\left[\frac{\pi}{3}+\sin^{-1}\left(\frac{1}{2}\right)\right]=\sin\left[\frac{\pi}{3}+\frac{\pi}{6}\right]=\sin\left(\frac{\pi}{2}\right)=1$$

21. Ans. (d): We have, $\sin(\tan^{-1}x)$

Let

$$\tan^{-1}x=\theta\Rightarrow x=\tan\theta\Rightarrow \sin\theta=\frac{x}{\sqrt{x^2+1}}$$

$$\therefore \sin(\tan^{-1}x)=\sin\theta=\frac{x}{\sqrt{x^2+1}}$$

22. Ans. (a): We have,

$$\tan^{-1}\left(\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right),\pi<x<\frac{3\pi}{2}$$

$$=\tan^{-1}\left(\frac{\left|\sqrt2\cos\frac{x}{2}\right|+\left|\sqrt2\sin\frac{x}{2}\right|}{\left|\sqrt2\cos\frac{x}{2}\right|-\left|\sqrt2\sin\frac{x}{2}\right|}\right)$$

$$=\tan^{-1}\left(\frac{-\sqrt2\cos\frac{x}{2}+\sqrt2\sin\frac{x}{2}}{-\sqrt2\cos\frac{x}{2}-\sqrt2\sin\frac{x}{2}}\right)\left(\because\frac{\pi}{2}<\frac{x}{2}<\frac{3\pi}{4}\right)$$

$$=\tan^{-1}\left(\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right)=\tan^{-1}\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right)$$

$$=\tan^{-1}\left(\tan\left(\frac{x}{4}-\frac{x}{2}\right)\right)=\frac{\pi}{4}-\frac{x}{2}$$

23. Ans. (c): Range of $\tan^{-1} x = \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \frac{-\pi}{2} < y < \frac{\pi}{2}$

24. Ans. (a): We have,

$\sin\left[\frac{\pi}{3} + \sin\left(\frac{1}{2}\right)\right] = \sin\left[\frac{\pi}{3} + \sin^{-1}\left(\sin\frac{\pi}{6}\right)\right] = \sin\left[\frac{\pi}{3} + \frac{\pi}{6}\right]$

$= \sin\left(\frac{\pi}{2}\right) = 1$

25. Ans. (d) : f(x) = |cos x|

At $\frac{\pi}{2} < x < \pi$, $\cos x < 0$

$\therefore |\cos x| = -\cos x \Rightarrow f(x) = -\cos x$

$\therefore f\left(\frac{3\pi}{4}\right) = -\cos\left(\frac{3\pi}{4}\right) = -\cos\left(\pi - \frac{\pi}{4}\right)$

$= \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $[\because \cos(\pi-\theta) = -\cos\theta]$

26. Ans. (d) : All trigonometric functions are periodic and hence not invertible over their respective domains but all trigonometric functions have inverse over their restricted domains.

Inverse of $\tan^{-1}x$ is tanx which is defined for

$x \in R - (2n+1)\frac{\pi}{2}, n \in Z$

$\therefore$ Assertion is false and reason is true.

27. Ans. (b): We have,

$\sin^{-1}\left(\cos\frac{13\pi}{5}\right) = \sin^{-1}\left[\cos\left(2\pi + \frac{3\pi}{5}\right)\right]$

$= \sin^{-1}\left[\cos\frac{3\pi}{5}\right] = \sin^{-1}\left[\cos\left(\frac{\pi}{2} + \frac{\pi}{10}\right)\right]$

$= \sin^{-1}\left(-\sin\frac{\pi}{10}\right) = -\sin^{-1}\left(\sin\frac{\pi}{10}\right) = -\frac{\pi}{10}$

28. Ans. (d): We know that $\cot^{-1}(x) \in (0, \pi)$

$\cot^{-1}(-\sqrt{3}) = \cot^{-1}\left(-\cos\frac{\pi}{6}\right)$

$= \cot^{-1}\left[\cot\left(\pi - \frac{\pi}{6}\right)\right]$

$[\because \cot(\pi-\theta) = -\cot\theta]$

$= \cot^{-1}\left[\cot\left(\frac{5\pi}{6}\right)\right] = \frac{5\pi}{6}$ $[\because \cot^{-1}[\cot\theta] = \theta]$

Thus, the principal value of $\cot^{-1}(-\sqrt{3})$ is $\frac{5\pi}{6}$.

29. Ans. (c): Given, $\tan^{-1}3 + \tan^{-1}\lambda = \tan^{-1}\left(\frac{3+\lambda}{1-3\lambda}\right)$

$\tan^{-1}3 + \tan^{-1}\lambda = \tan^{-1}\left(\frac{3+\lambda}{1-3\lambda}\right)$ for

$3\lambda < 1 \therefore 3\lambda < 1 \Rightarrow \lambda < \frac{1}{3}$

30. Ans. (b): We have, $\tan^{-1}\left(\tan\frac{3\pi}{5}\right)$

We know that the range of $\tan^{-1} x$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \tan^{-1}\left(\tan\frac{3\pi}{5}\right) = \tan^{-1}\left(\tan\left(\pi - \frac{2\pi}{5}\right)\right)$

$= \tan^{-1}\left[-\tan\left(\frac{2\pi}{5}\right)\right]$ $[\because \tan(\pi-\theta) = \tan\theta]$

$= -\tan\theta^{-1}\left[\tan\left(\frac{2\pi}{5}\right)\right] = -\frac{2\pi}{5}$

$[\because \tan^{-1}(\tan\theta) = 0]$

➢ Assertion-Reasoning (1 mark)

31. Sol. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation: Both A and R are true and R is the correct explanation of A.

32. Sol. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation: Both A and R are true and R is the correct explanation of A.

33. Sol. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation: Both A and R are true and R is the correct explanation of A.

34. Sol. **(d)** A is false but R is true.

Explanation: Assertion- $sin^{-1}\ x$ should not be confused with $(sinx)^{-1}$. In fact $(sinx)^{-1} = \frac{1}{\sin x}$ = and similarly for trigonometric functions.

Reason- The value of an inverse trigonometric function that lies in the range of the principal branch, is called the principal value of that inverse trigonometric function. Hence, we can say that Assertion is false and Reason is true.

➢ Case Study Question

35. Sol.

(i) **(a)** $\tan^{-1} 1$

Explanation: $\tan^{-1} 1$

(ii) **(d)** 41.5 km/hr

Explanation: 41.5 km/hr

(iii) **(b)** $\tan^{-1}\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$

Explanation: $\tan^{-1}\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$

(iv) **(a)** $\tan^{-1}\frac{1}{\sqrt{3}}$

Explanation: $\tan^{-1}\frac{1}{\sqrt{3}}$

(v) **(b)** $\tan^{-1}\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$

Explanation: $\tan^{-1}\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$

36. Sol.

(i) **(d)** $\tan^{-1}(\frac{1}{2})$

Explanation: $\tan^{-1}(\frac{1}{2})$

(ii) **(c)** $\tan^{-1}(\frac{4}{3})$

Explanation: $\tan^{-1}(\frac{4}{3})$

(iii) **(b)** $\tan^{-1}(\frac{11}{2})$

Explanation: $\tan^{-1}(\frac{11}{2})$

(iv) **(b)** $\tan^{-1}(\frac{1}{8})$

Explanation: $\tan^{-1}(\frac{1}{8})$

(v) **(b)** R, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Explanation: R, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

➢ Questions

37. Ans. $$\cos^{-1}\left[\cos\left(-\frac{7\pi}{3}\right)\right] = \cos^{-1}\left[\cos\left(\frac{7\pi}{3}\right)\right]$$

$$(\because \cos(-\theta) = \cos\theta)$$

$$= \cos^{-1}\left(\cos\left(2\pi + \frac{\pi}{3}\right)\right) = \cos^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right)$$

$$= \frac{\pi}{3} (\because \cos^{-1}(\cos x) = x\ \forall 0 \le x \le \pi)$$

38. Ans. Consider L.H.S.

$$= \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$= \frac{9}{4}\cos^{-1}\left(\frac{1}{3}\right) \text{........(i)}$$

$$\left(\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right)$$

Let $a = \cos^{-1}\left(\frac{1}{3}\right)$

$$= \cos a = \frac{1}{3} \Rightarrow \sin a = \sqrt{1-\cos^2 a}$$

$$[\because \sin^2\theta + \cos^2\theta = 1]$$

$$\Rightarrow \sin a = \sqrt{1-\left(\frac{1}{3}\right)^2} = \sqrt{1-\frac{1}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

$$\Rightarrow a = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

So, L.H.S. = $\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = R.H.S.$

39. Ans. Consider, L.H.S. = $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$

$$= \tan^{-1}\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{\left(\frac{4}{3}+\frac{1}{7}\right)}{1-\frac{4}{21}} = \tan^{-1}\frac{\frac{31}{21}}{\frac{17}{21}}$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$= \tan^{-1}\frac{31}{17} = R.H.S.$$

Hence proved.

40. Ans. Let $x=\cos^{-1}\left(\frac{12}{13}\right) and\ y=\sin^{-1}\left(\frac{3}{5}\right)$

Or $\cos x=\frac{12}{13} and \sin y=\frac{3}{5}$

Now, $\sin x=\sqrt{1-\cos^2 x}\ and\ \cos y=\sqrt{1-\sin^2 y}$

$\Rightarrow \sin x=\sqrt{1-\frac{144}{169}}\ and\ \cos y=\sqrt{1-\frac{9}{25}}$

$\Rightarrow \sin x=\frac{5}{13}\ and\ \cos y=\frac{4}{5}$

We know that,

$\sin(x+y)=\frac{5}{13}\times\frac{4}{5}+\frac{12}{13}\times\frac{3}{5}=\frac{20}{65}+\frac{36}{65}=\frac{56}{65}$

$\Rightarrow x+y=\sin^{-1}\left(\frac{56}{65}\right)$

Or, $\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{56}{65}\right)$

41. Ans. Consider L.H.S. = $\tan^{-1}\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right)$

Put $x=\cos\theta$, we get

L.H.S. = $\tan^{-1}\left[\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right]$

$=\tan^{-1}\left[\frac{\left|\sqrt{2}\cos\frac{\theta}{2}\right|+\left|\sqrt{2}\sin\frac{\theta}{2}\right|}{\left|\sqrt{2}\cos\frac{\theta}{2}\right|-\left|\sqrt{2}\sin\frac{\theta}{2}\right|}\right]$

$[\because 1+\cos^2\theta=2\cos^2\theta, 1-\cos^2\theta=2\sin^2\theta]$

$=\tan^{-1}\left[\frac{-\sqrt{2}\cos\frac{\theta}{2}+\sqrt{2}\sin\frac{\theta}{2}}{-\sqrt{2}\cos\frac{\theta}{2}-\sqrt{2}\sin\frac{\theta}{2}}\right]=\tan^{-1}\left[\frac{-\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{-\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right]$

42. Soln.

We define,

$\cos^{-1}x:[-1,1]\to[0,\pi]$

For every $x\in[-1,1]$ there exists a unique $\theta\in[0,\pi]$ such that $\cos\theta=x$. Thus θ is called a principal value of $\cos^{-1}x$. However we could also define.

$\cos^{-1}x:[-1,1]\to[-\pi,0]$

or $\cos^{-1}x:[-1,1]\to[\pi,2\pi]$

Such that $\cos\theta=x$, Infact we can find infinitely many such intervals. But only the value of θ lying in $[0,\pi]$ is called the principal value.

That is, the principal value of $\cos^{-1}x$ is the numerically least among all the values of $\cos^{-1}x$.

The principal value branch of $\cos^{-1}x$ is $[0,\pi]$.

43. Soln. We know that the range of principal value branch of $\cos^{-1}$ and $\sin^{-1}$ are $[0,\pi]$ and $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ respectively.

Let $\cos^{-1}\left(\frac{1}{2}\right)=x\Rightarrow\frac{1}{2}=\cos x$

Then, $\frac{1}{2}=\cos\left(\frac{\pi}{3}\right), where\ \frac{\pi}{3}\in[0,\pi]$

Let $\sin^{-1}\left(\frac{1}{2}\right)=y\Rightarrow\frac{1}{2}=\sin y$

Then, $\frac{1}{2}=\sin\left(\frac{\pi}{6}\right), where\ \frac{\pi}{6}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\therefore \cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}+2.\frac{\pi}{6}=\frac{2\pi}{3}$

44. Soln.

$\sin:R\to R$ such that

$\sin\theta=x$ for all $\theta\in R$ is a many – one into function.

This function cannot have an inverse.

Therefore, we restrict the domain to the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and codomain to the interval [-1, 1].

Then,

$\sin:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to[-1,1]$ is a one – one, onto function and is therefore, invertible.

The inverse of the sine function is defined as

$\sin^{-1}:[-1,1]\to\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Such that $\sin^{-1}x=\theta$

$\Leftrightarrow \sin\theta=x$

Thus, the domain of the function is [-1, 1] and the range is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

45. Soln. Given equation is $2\cos^{-1}x+\sin^{-1}x=\frac{11\pi}{6}$

$$\Rightarrow \cos^{-1}x+(\cos^{-1}x+\sin^{-1}x)=\frac{11\pi}{6}$$

$$\Rightarrow \cos^{-1}x+\frac{\pi}{2}=\frac{11\pi}{6}\left(Given\ \cos^{-1}x+\sin^{-1}x=\frac{\pi}{2}\right)$$

$$\Rightarrow \cos^{-1}x=\frac{4\pi}{3}$$

Which is not possible as $\cos^{-1}x\in[0,\pi]$.

Thus, given equation has no solution.

46. Soln. We have,

$$\cos\left\{\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right\}$$

Now let

$$\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)=\theta$$

$$\Rightarrow \quad \cos\theta=\frac{-\sqrt{3}}{2}$$

$$=-\cos\frac{\pi}{6}$$

$$=\cos\left(\pi-\frac{\pi}{6}\right)$$

$$=\cos\frac{5\pi}{6}$$

$$\Rightarrow \quad \theta=\frac{5\pi}{6}$$

$$\therefore \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)=\frac{5\pi}{6}$$

$$\therefore \cos\left\{\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right\}=\cos\left(\frac{5\pi}{6}+\frac{\pi}{6}\right)$$

$$=\cos\pi$$

$$=-1$$

47. Soln.

Here, $\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2=1$

Put $\frac{3}{5}=\cos\theta\ and\ \frac{4}{5}=\sin\theta\Rightarrow\tan\theta=\frac{4}{3}$

$$\Rightarrow \theta=\tan^{-1}\left(\frac{4}{3}\right)$$

$$\therefore \cos^{-1}\left(\frac{3}{5}\cos x+\frac{4}{5}\sin x\right)$$

$$=\cos^{-1}(\cos\theta\cos x+\sin\theta\sin x)$$

$$=\cos^{-1}\{\cos(x-\theta)\}=x-\theta=x-\tan^{-1}\left(\frac{4}{3}\right)$$

48. Soln. Let $x=a\cos\theta$

$$\therefore \tan^{-1}\sqrt{\frac{a-x}{a+x}}=\tan^{-1}\sqrt{\frac{a-a\cos\theta}{a+a\cos\theta}}$$

$$=\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$=\tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}$$

$$=\tan^{-1}\sqrt{\tan^2\frac{\theta}{2}}$$

$$=\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$=\frac{\theta}{2}$$

$$\because x=a\cos\theta$$

$$\Rightarrow \quad \cos\theta=\frac{x}{a}$$

$$\Rightarrow \quad \theta=\cos^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \quad \tan^{-1}\sqrt{\frac{a-x}{a+x}}=\frac{\theta}{2}$$

$$=\frac{1}{2}\cos^{-1}\frac{x}{a}$$

49. Soln. We have, $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$

$$\because -\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2},\frac{-\pi}{2}\le\sin^{-1}y\le\frac{\pi}{2}$$

$$and\ \frac{-\pi}{2}\le\sin^{-1}z\le\frac{\pi}{2}$$

$\therefore$ The above condition will true if

$$\sin^{-1}x=\sin^{-1}y=\sin^{-1}z=\frac{\pi}{2}\Rightarrow x=y=z=1$$

Thus, there is only one triplet.

50. Soln.

Let $\cot^{-1}\frac{3}{4}=\theta \Rightarrow \cot\theta=\frac{3}{4}$

$\therefore \ \operatorname{cosec}\theta=\sqrt{1+\cot^2\theta}=\sqrt{1+\frac{9}{16}}=\frac{5}{4}$

$\therefore \sin\theta=\frac{4}{5}\Rightarrow \theta=\sin^{-1}\frac{4}{5}$

So, $\sin\left(\cot^{-1}\frac{3}{4}\right)=\sin\left(\sin^{-1}\frac{4}{5}\right)=\frac{4}{5}$

Let $\tan^{-1}x=\phi$. Then, $\tan\phi=x$.

$\therefore \ \sec\phi=\sqrt{1+\tan^2\phi}=\sqrt{1+x^2}$

$\therefore \ \cos\phi=\frac{1}{\sqrt{1+x^2}}$

So, $\cos(\tan^{-1}x)=\cos\phi=\frac{1}{\sqrt{1+x^2}}$

Thus, $\frac{1}{\sqrt{1+x^2}}=\frac{4}{5}\Rightarrow\frac{1}{1+x^2}=\frac{16}{25}\Rightarrow 16x^2=9$

$\Rightarrow x^2=\frac{9}{16}\Rightarrow x=\pm\frac{3}{4}$.

51. Soln.

$\sin^{-1}\left(\cos\left(8\pi+\frac{3\pi}{5}\right)\right)=\sin^{-1}\left(\cos\frac{3\pi}{5}\right)$

$=\sin^{-1}\left(\sin\left(\frac{\pi}{2}-\frac{3\pi}{5}\right)\right)$

$=\sin^{-1}\left(\sin\left(-\frac{\pi}{10}\right)\right)$

$=-\frac{\pi}{10} \qquad \left[\because -\frac{\pi}{10}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right]$

52. Soln.

We have, $\cos^{-1}\left(\frac{1}{2}\right)=\cos^{-1}\left(\cos\frac{\pi}{3}\right)$

$=\frac{\pi}{3} \qquad \left[\because \frac{\pi}{3}\in[0,\pi]\right]$

Also $\sin^{-1}\left(\frac{1}{2}\right)=\sin^{-1}\left(\sin\frac{\pi}{6}\right)$

$=\frac{\pi}{6} \qquad \left[\because \frac{\pi}{6}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right]$

$\therefore \cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}+2\left(\frac{\pi}{6}\right)$

$=\frac{\pi}{3}+\frac{\pi}{3}=\frac{2\pi}{3}$

53. Soln. Let $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$

$\Rightarrow \tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]=\tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right]$

$=\tan^{-1}\left[\frac{\sqrt{\sec^2\theta}-1}{\tan\theta}\right]=\tan^{-1}\left[\frac{\sec\theta-1}{\tan\theta}\right]$

$=\tan^{-1}\left[\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right]=\tan^{-1}\left[\frac{1-\cos\theta}{\cos\theta}\times\frac{\cos\theta}{\sin\theta}\right]$

$=\tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}\right]=\tan^{-1}\left(\tan\frac{\theta}{2}\right)$

$=\frac{\theta}{2}=\frac{1}{2}(\tan^{-1}x)$

54. Soln.

We have, $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$

$=\tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$

$=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}-x\right)\right\}=\frac{\pi}{4}-x$

55. Soln.

$\sin\left(2\sin^{-1}\frac{3}{5}\right)=\sin\left(2\tan^{-1}\frac{3}{4}\right)$

$$= \sin\left[\tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\frac{9}{16}}\right)\right]$$

$$= \sin\left[\tan^{-1}\left(\frac{3}{2}\times\frac{16}{7}\right)\right]$$

$$= \sin\left[\tan^{-1}\frac{24}{7}\right]$$

$$= \sin\left[\sin^{-1}\frac{24}{25}\right]$$

$$= \frac{24}{25}$$

$$\left[\because \sin^{-1}(\sin\theta) = \theta \,\forall\, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$$

56. Soln.

Since, $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$

$$= \cos\left[\cos^{-1}\left(\cos\frac{5\pi}{6}\right)+\frac{\pi}{6}\right]$$

$$\left[\because \cos\frac{5\pi}{6} = \cos\left(\pi-\frac{\pi}{6}\right) = \frac{-\sqrt{3}}{2}\right]$$

$$= \cos\left(\frac{5\pi}{6}+\frac{\pi}{6}\right) \qquad \{\because \cos^{-1}(\cos\theta) = \theta; \theta \in [0,\pi]\}$$

$$= \cos(\pi) = -1$$

57. Soln.

$$\sec^2(\tan^{-1}2) + \text{cosec}^2(\cot^{-1}3)$$

$$= [\sec(\tan^{-1}2)]^2 + [\text{cosec}(\cot^{-1}3)]^2$$

$$= [\sec(\sec^{-1}\sqrt{5})]^2 + [\text{cosec}(\text{cosec}^{-1}\sqrt{10})]^2$$

$$= \left(\sqrt{5}\right)^2 + \left(\sqrt{10}\right)^2 = 5+10 = 15$$

58. Soln.

Given, $f(x) = \sin^{-1}x + \tan^{-1}x + \sec^{-1}x$

Domain of $\sin^{-1}x = [-1,1]$

Domain of $\tan^{-1}x = (-\infty,\infty)$

Domain of $\sec^{-1}x = (-\infty,\infty)-(-1,1)$

Domain of

$$f(x) = [-1,1] \cap (-\infty,\infty) \cap [(-\infty,\infty)-(-1,1)]$$
$$= \{-1,1\}$$

Now, $f(-1) = \sin^{-1}(-1) + \tan^{-1}(-1) + \sec^{-1}(-1)$

$$= -\frac{\pi}{2} - \frac{\pi}{4} + \pi = \frac{\pi}{4}$$

And $f(1) = \sin^{-1}(1) + \tan^{-1}(1) + \sec^{-1}(1)$

$$= \frac{\pi}{2} + \frac{\pi}{4} + 0 = \frac{3\pi}{4}$$

Range of $f(x) = \left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$

59. Soln.

We know that the minimum value of $\text{cosec}^{-1}x$ is $-\frac{\pi}{2}$ which is attained at x = -1.

$$\therefore \text{cosec}^{-1}x + \text{cosec}^{-1}y + \text{cosec}^{-1}z = -\frac{3\pi}{2}$$

$$\Rightarrow \text{cosec}^{-1}x + \text{cosec}^{-1}y + \text{cosec}^{-1}z = \left(-\frac{\pi}{2}\right)+\left(-\frac{\pi}{2}\right)+\left(-\frac{\pi}{2}\right)$$

$$\Rightarrow \text{cosec}^{-1}x = -\frac{\pi}{2}, \text{cosec}^{-1}y = -\frac{\pi}{2}, \text{cosec}^{-1}z = -\frac{\pi}{2}$$

$$\Rightarrow x=-1,\ y=-1,\ z=-1$$

$$\therefore \frac{x}{y}+\frac{y}{z}+\frac{z}{x} = \frac{(-1)}{(-1)}+\frac{(-1)}{(-1)}+\frac{(-1)}{(-1)} = 3$$

60. Soln.

Let $\cos^{-1}x = \theta$, then $\cos\theta = x$, where $\theta \in [0,\pi]$

$$\therefore \tan(\cos^{-1}x) = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \frac{\sqrt{1-x^2}}{x}$$

$$Hence, \tan\left(\cos^{-1}\left(\frac{8}{17}\right)\right) = \frac{\sqrt{1-(8/17)^2}}{8/17} = \frac{15}{8}$$

61. Soln.

As we know, $\tan^{-1}(\tan x) = x; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

And $\cos^{-1}(\cos x) = x;\ x \in [0,\pi]$

$$\therefore \tan^{-1}\left(\tan\frac{5\pi}{6}\right) + \cos^{-1}\left(\cos\frac{13\pi}{6}\right)$$

$$= \tan^{-1}\left[\tan\left(\pi-\frac{\pi}{6}\right)\right] + \cos^{-1}\left[\cos\left(\pi+\frac{7\pi}{6}\right)\right]$$

$$\left\{\because \frac{5\pi}{6} \notin \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) and\ \frac{13\pi}{6} \notin [0,\pi]\right\}$$

$$= \tan^{-1}\left(-\tan\frac{\pi}{6}\right) + \cos^{-1}\left(-\cos\frac{7\pi}{6}\right)$$

$$[\because \tan(\pi-\theta) = -\tan\theta \text{ and } \cos(\pi+\theta) = -\cos\theta]$$

$$= -\tan^{-1}\left(\tan\frac{\pi}{6}\right) + \pi - \left[\cos^{-1}\cos\left(\frac{7\pi}{6}\right)\right]$$

$$\{\because \tan^{-1}(-x) = -\tan^{-1}x; x \in R \text{ and } \cos^{-1}(-x) = \pi - \cos^{-1}x; x \in [-1,1]\}$$

$$= -\tan^{-1}\left(\tan\frac{\pi}{6}\right) + \pi - \cos^{-1}\left[\cos\left(\pi + \frac{\pi}{6}\right)\right]$$

$$= -\tan^{-1}\left(\tan\frac{\pi}{6}\right) + \pi - \left[\cos^{-1}\left(-\cos\frac{\pi}{6}\right)\right]$$

$$= -\tan^{-1}\left(\tan\frac{\pi}{6}\right) + \pi - \pi + \cos^{-1}\left(\cos\frac{\pi}{6}\right)$$

$$= -\frac{\pi}{6} + 0 + \frac{\pi}{6} = 0$$

62. Soln.

Since, $\sin^{-1}\left(\sin\frac{7\pi}{6}\right)$

$$= \sin^{-1}\left(\sin\left(\pi + \frac{\pi}{6}\right)\right) = \sin^{-1}\left(-\sin\frac{\pi}{6}\right)$$

$$= \sin^{-1}\left(\sin\left(\frac{-\pi}{6}\right)\right) = \frac{-\pi}{6}$$

$$\cos^{-1}\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}, \tan^{-1}\left(\tan\frac{5\pi}{4}\right)$$

$$= \tan^{-1}\tan\left(\pi + \frac{\pi}{4}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}\right)\right] = \frac{\pi}{4} \text{ and } \cot^{-1}\left\{\cot\left(-\frac{\pi}{4}\right)\right\}$$

$$= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

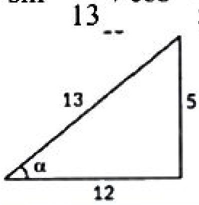
Hence, required value is

$$\frac{1}{\pi}\left\{216\times -\frac{\pi}{6} + 27\times\frac{2\pi}{3} + 28\times\frac{\pi}{4} + 200\times\frac{3\pi}{4}\right\}$$

$$= -36 + 18 + 7 + 150 = 139$$

63. Soln. We have to prove,

$$\sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5} = \tan^{-1}\frac{63}{16}$$



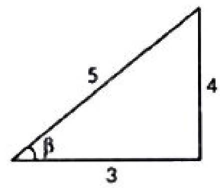
Consider, $\sin^{-1}\frac{5}{13} = \alpha$

$$\Rightarrow \sin\alpha = \frac{5}{13} \text{ and } \cos\alpha = \frac{12}{13}$$

$$\therefore \quad \tan\alpha = \frac{5}{12} \quad \ldots\ldots\ldots\ldots(i)$$

Again, consider $\cos^{-1}\frac{3}{5} = \beta \Rightarrow \cos\beta = \frac{3}{5}$

$$\sin\beta = \frac{4}{5}, \tan\beta = \frac{4}{3}$$



As we know,

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha.\tan\beta}$$

$$\Rightarrow \tan(\alpha+\beta) = \frac{\frac{5}{12} + \frac{4}{3}}{1 - \frac{5}{12}.\frac{4}{3}}$$

$$\Rightarrow \tan(\alpha+\beta) = \frac{\frac{15+48}{36}}{\frac{36-20}{36}}$$

$$\Rightarrow \tan(\alpha+\beta) = \frac{63}{16} \Rightarrow \alpha + \beta = \tan^{-1}\frac{63}{16}$$

$$\Rightarrow \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5} = \tan^{-1}\frac{63}{16}$$

64. Soln.

Given, $f(x) = \sin^{-1}\sqrt{x-1}$

Since, $x - 1 \geq 0$ *and* $-1 \leq \sqrt{x-1} \leq 1$

$\therefore\ 0 \leq x - 1 \leq 1 \Rightarrow 1 \leq x \leq 2$

65. Soln.

We have given, $f(x) = \cos^{-1}(2x-1)$

Since, $-1 \leq 2x - 1 \leq 1$

$\Rightarrow \quad 0 \leq 2x \leq 2$

$\Rightarrow \quad 0 \leq x \leq 1$

66. Soln.

Since, $\cos^{-1}\left(\cos\frac{14\pi}{3}\right)=\cos^{-1}\cos\left(4\pi+\frac{2\pi}{3}\right)$

$=\cos^{-1}\cos\frac{2\pi}{3}=\frac{2\pi}{3}$

$\left\{\because \cos^{-1}(\cos x)=x, x\in[0,\pi]\right\}$

SURE SHOT QUESTIONS

ARVIND ACADEMY

Chapter – 03 (Solution)

Matrices

MCQ (1 mark)

1. Soln. (a): $P=\begin{bmatrix}0&0&4\\0&4&0\\4&0&0\end{bmatrix}$

P is a 3 x 3 matrix

Number of rows = number of columns

Hence, P is a square matrix.

2. Soln. (d): Required number of matrices = 2^9 = 512

3. Soln. (b): $\begin{bmatrix}2x+y&4x\\5x-7&4x\end{bmatrix}=\begin{bmatrix}7&7y-13\\y&x+6\end{bmatrix}$

On comparing, we get

$4x=x+6\Rightarrow x=2$ and $2x+y=7\Rightarrow y=7-4=3$

4. Soln. (d): A is of order 3 x m and B is of order 3 x n and m = n.

So, 5A – 2B is of order 3 x m or 3 x n.

5. Soln. (d): $A=\begin{bmatrix}0&1\\1&0\end{bmatrix}$

$$A^2\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}0+1&0+0\\0+0&1+0\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

6. Soln. (a): $a_{11}=0,\ a_{12}=1,\ a_{21}=1,\ a_{22}=0$

$\therefore\quad A=\begin{bmatrix}0&1\\1&0\end{bmatrix}$

$$\therefore\quad A^2=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}0+1&0+0\\0+0&1+0\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I$$

7. Soln. (b): Let

$$A=\begin{bmatrix}1&0&0\\0&2&0\\0&0&0\end{bmatrix}\Rightarrow A^T=\begin{bmatrix}1&0&0\\0&2&0\\0&0&0\end{bmatrix}=A$$

$\therefore$ A is a symmetric matrix

8. Soln. (c): Let $A=\begin{bmatrix}0&-5&8\\5&0&12\\-8&-12&0\end{bmatrix}$

$$A^T=\begin{bmatrix}0&5&-8\\-5&0&-12\\8&12&0\end{bmatrix}=-\begin{bmatrix}0&-5&8\\5&0&12\\-8&-12&0\end{bmatrix}=-A$$

Since, $A^T=-A$ $\therefore$ A is a skew symmetric matrix.

9. Soln. (d): Let matrix B is of order $p\times q$.

$\therefore$ Matrix B' is of order $q\times p$.

Matrix A is of order $m\times n$.

Since, AB' is defined.

$\therefore$ Number of columns of A = number of rows of B'

$\Rightarrow\ n=q$

Also, B'A is defined

$\therefore$ number of column of B' = number of rows of A

$\Rightarrow\ p=m$

Hence, B is of order $p\times q$ i.e., $m\times n$.

10. Soln. (a): $(AB'-BA')'=(AB')'-(BA')'$

$=(B')'A'-(A')'B'=BA'-AB'=-(AB'-BA')$

Hence, $(AB' - BA')$ is a skew symmetric matrix.

11. Soln. (a): We have, $A^2 = I$

Now, $(A-I)^3 + (A+I)^3 - 7A$

$= A^3 - I^3 - 3A^2I + 3AI^2 + A^3 + I^3 + 3A^2I + 3AI^2 - 7A$

$= 2A^3 + 6AI^2 - 7A = 2A^2A + 6AI + 7A$

$= 2IA + 6A - 7A = 2A + 6A - 7A = A \qquad [\because A^2 = I]$

12. Soln. (d)

13. Soln. (d): $\begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$

Applying $C_2 \to C_2 - 2C_1$, we get

$\begin{bmatrix} 1 & -5 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 2 & 0 \end{bmatrix}$

14. Ans. (c) : We have, $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} x & 0 \\ 1 & 1 \end{bmatrix}$

$\therefore \ B^2 = \begin{bmatrix} x & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} x^2 & 0 \\ x+1 & 1 \end{bmatrix}$

Now, it is given that $A = B^2$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} x^2 & 0 \\ x+1 & 1 \end{bmatrix}$

On comparing, we get

$x^2 = 1 \ and \ x+1 = 2 \Rightarrow x = \pm 1 \ and \ x = 1$

$\therefore \ x = 1$

15. Ans. (d): $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$

$\therefore \ x + y + z = 6$ ....(i)

$y + z = 3$ ....(ii)

$z = 2$ ....(iii)

$\Rightarrow \ y + 2 = 3$ [Using (ii) and (iii)]

$\Rightarrow \ y = 1$ ....(iv)

$\Rightarrow \ x + 1 + 2 = 6$ [Using (i), (iii) & (iv)]

$\Rightarrow \ x = 3$

So, 2x + y − z = (2 x 3) + 1 − 2 = 6 + 1 − 2 = 5

16. Ans. (b): We have, $x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$

$\begin{bmatrix} x + 2y \\ 2x + 5y \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$

$\Rightarrow \ x + 2y = 4$ ....(i) and $2x + 5y = 9$ ....(ii)

Solving (i) and (ii), we get x = 2, y = 1

17. Ans. (a): Given that $A^2 = A$

Consider $(I + A)^2 - 3A$

$= I^2 + A^2 + 2AI - 3A$

$= I + A + 2A - 3A$

$[\because I^2 = I, \ A^2 = A(\text{given})]$

18. Ans. (d)

19. Ans. (a): Consider C(A + B′) i.e., $C_{3\times3}(A_{2\times3} + B'_{2\times3})$

$= C_{3\times3}(A + B')_{2\times3}$

Here, number of columns in the matrix C is 3 and number of rows in the matrix (A + B′) is 2. So, it is not defined.

20. Ans. (d)

21. Ans. (a): We have, $A^2 = A$

Now, $(I - A)^3 + A = (I - A)(I - A)(I - A) + A$

$= (I.I - I.A - A.I + A.A)(I - A) + A$

$= (I - A - A + A)(I - A) + A$

$[\because I.A = A.I = A \text{ and } A^2 = A]$

$= (I - A)(I - A) + A$

$= (I.I - I.A - A.I + A.A) + A$

$= (I - A - A + A) + A = (I - A) + A = I$

22. Ans. (a): Consider, AB = [2 -3 4] $\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} = [6 - 6 + 8] = [8]$

And XY = [1 2 3] $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = [2 + 6 + 12] = [20]$

$$AB + XY = [8]+[20]=[28]$$

23. Ans. (c): We have, $A^2 - A + I = O$

Pre – multiplying with A^{-1} on both sides, we get

$$(A^{-1}A).A.A^{-1}.A + A^{-1}I = A^{-1}.O$$
$$\Rightarrow I.A - I + A^{-1} = O$$
$$\Rightarrow A^{-1} = -(A - I) = I - A$$

24. Ans. (c): In a skew-symmetric matrix, the (i, j)th element is negative of the $(j,i)^{th}$ element. Hence, the $(i,i)^{th}$ element = 0.

25. Ans. (a): From the definition of equality of two matrices, we have

2a + b = 4 ....(i) a – 2b = -3 ......(ii)

5c – d = 11 ......(iii) 4c + 3d = 24 ....(iv)

Solving (i) and (ii), we get

$5a = 5 \Rightarrow a = 1,\ b = 2$

Solving (iii) and (iv), we get

$19c = 57 \Rightarrow c = 3,\ d = 4$

$\therefore\ a + b - c + 2d = 1 + 2 - 3 + 8 = 8$

26. Ans. (b): We know that the sum of two matrices is defined only if both matrices have same order.

Here 5A + 3B is defined if A and B have same order.

$\Rightarrow 3 \times n = m \times 5 \Rightarrow n = 5,\ m = 3$

So, order of matrix C is 3×5 and $m \neq n$

27. Ans. (d): We have, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\therefore\ A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+1 & 0+0 \\ 0+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

28. Ans. (b): We have,

$$A = \begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix} \Rightarrow kA = \begin{bmatrix} 0 & 2k \\ 3k & -4k \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix} = \begin{bmatrix} 0 & 2k \\ 3k & -4k \end{bmatrix} \text{(Given)}$$
$$\Rightarrow -4k = 24,\ 3a = 2k, 2b = 3k$$
$$\Rightarrow k = -6,\ a = -4,\ b = -9$$

29. Ans. (d): We have, $(I + A)^3 - 7A$

$$= I^3 + A^3 + 3I^2A + 3IA^2 - 7A$$
$$= I + A.A + 3A + 3A - 7A \quad (\because A^2 = A)$$
$$= I + A + 3A + 3A - 7A = I$$

30. Ans. (c): We have, $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$

$$\Rightarrow A^2 = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} = \begin{bmatrix} \alpha^2 + \beta\gamma & 0 \\ 0 & \gamma\beta + \alpha^2 \end{bmatrix}$$

But $A^2 = 3I$

$$\Rightarrow \begin{bmatrix} \alpha^2 + \beta\gamma & 0 \\ 0 & \alpha^2 + \beta\gamma \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$
$$\Rightarrow \alpha^2 + \beta\gamma = 3$$
$$\Rightarrow 3 - \alpha^2 - \beta\gamma = 0$$

31. Ans. (c): We know that if A and B are non – singular matrices of same order, then

$$(AB)^{-1} = B^{-1}A^{-1}; (AB^{-1})^{-1} = (B^{-1})^{-1}A^{-1} = BA^{-1}$$

32. Ans. (d): We have,

$$AB = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 2+4+0 & 2-2+0 & -4+4+0 \\ 4-12+8 & 4+6-4 & -8-12+20 \\ 0-4+4 & 0+2-2 & 0-4+10 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6I \Rightarrow B^{-1} = \frac{1}{6}A$$

33. Ans. (b): We have A = A^T

$$\Rightarrow \begin{bmatrix} 5 & x \\ y & 0 \end{bmatrix} = \begin{bmatrix} 5 & y \\ x & 0 \end{bmatrix}$$

On comparing, we get x = y.

34. Ans. (d): A = [1 2 3]

$$A' = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

So, $AA' = [1\ 2\ 3]\begin{bmatrix}1\\2\\3\end{bmatrix} = [1+4+9] = [14]$

35. Ans. (b): We have, $P' = 2P + I$ ....(i)

Now, $(P')' = (2P + I)' = 2P' + I$

$\Rightarrow\ P = 2(2P + I) + I$ [Using (i)]

$\Rightarrow\ P = 4P + 3I \Rightarrow P = -I$

36. Ans. (b): We have, $A = \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$

And $A + A' = I$

$\Rightarrow \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix} + \begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$\Rightarrow \begin{bmatrix}2\cos\alpha & 0\\ 0 & 2\cos\alpha\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$\Rightarrow\ 2\cos\alpha = 1 \Rightarrow \cos\alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3}$

➢ Assertion-Reasoning (1 mark)

37. Sol. (a) Both A and R are true and R is the correct explanation of A.

Explanation: Both A and R are true and R is the correct explanation of A.

38. Sol.

(d) A is false but R is true.

Explanation: Assertion: Given, $A^2 = kA - 2I$

$\Rightarrow AA = kA - 2I$

$\Rightarrow \begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix}\begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix} = k\begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix} - 2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$\Rightarrow \begin{bmatrix}9-8 & -6+4\\ 12-8 & -8+4\end{bmatrix} = \begin{bmatrix}3k & -2k\\ 4k & -2k\end{bmatrix} - \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$

$\Rightarrow \begin{bmatrix}1 & -2\\ 4 & -4\end{bmatrix} = \begin{bmatrix}3k-2 & -2k\\ 4k & -2k-2\end{bmatrix}$

By definition of equality of matrix, the given matrices are equal and their corresponding elements are equal. Now, comparing the corresponding elements, we get

$3k - 2 = 1 \Rightarrow k = 1$

$\Rightarrow -2k = -2 = k = 1$

$\Rightarrow 4k = 4 \Rightarrow k = 1$

$\Rightarrow -4 = -2A - 2 \Rightarrow k = 1$

Hence, k = 1

Reason: We have,

$(A + B)(A + B) = A(A + B) + B(A + B)$

$= A^2 + AB + BA + B^2$

39. Sol. (c) A is true but R is false.

Explanation: Assertion: In general, the matrix A of order

2×2 is given by $A = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$

Now, $a_{ij} = i \times j$, i = 1, 2 and j = 1, 2

$\therefore a_{11} = 1, a_{12} = 2, a_{21} = 2$ and $a_{22} = 4$

Thus, matrix A is $\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$

Reason: If A is a 4×2 matrix, then A has $4 \times 2 = 8$ elements.

40. Sol.

(a) Both A and R are true and R is the correct explanation of A.

Explanation: We define $-A = (-1)A$

If $A = \begin{bmatrix}3 & 1\\ -5 & x\end{bmatrix}$

then $-A$ is given by

$-A = (-1)A = (-1)\begin{bmatrix}3 & 1\\ -5 & x\end{bmatrix} = \begin{bmatrix}-3 & -1\\ 5 & -x\end{bmatrix}$

➢ Case Study Question

41. Sol. (i) (b) A + B

Explanation: A + B

(ii) (c) 10000

Explanation: 10000

(iii) (b) A – B

Explanation: A - B

(iv) (b) ₹110, ₹200 and ₹120

Explanation: ₹110, ₹200 and ₹120

(v) (d) ₹1000, ₹600, ₹200

Explanation: ₹1000, ₹600, ₹200

42. Sol.

(i)

(b) $A = \text{Investment}\ \begin{array}{cc} X & Y\\ \end{array} [15000\ \ 20000]$; $B = \begin{array}{c} \\ X\\ Y\end{array}\begin{array}{c}\text{Invest rate}\\ \begin{bmatrix}0.1\\ 0.08\end{bmatrix}\end{array}$

Explanation: If ₹ 15000 is invested in bond X, then the amount invested in bond Y = ₹ (35000 - 15000) = ₹20000

$$A = \text{Investment}\begin{array}{c} \begin{array}{cc} X & Y \end{array} \\ \begin{bmatrix} 15000 & 20000 \end{bmatrix} \end{array}$$

$$\text{and } B = \begin{array}{c} \\ X \\ Y \end{array}\begin{array}{c} \text{Invest rate} \\ \begin{bmatrix} 10\,\% \\ 8\,\% \end{bmatrix} \end{array} = \begin{array}{c} \\ X \\ Y \end{array}\begin{array}{c} \text{Invest rate} \\ \begin{bmatrix} 0.1 \\ 0.08 \end{bmatrix} \end{array}$$

(ii) (a) Rs 3100
Explanation: The amount of interest received on each bond is given by

$$AB = [15000\ 20000]\times\begin{bmatrix} 0.1 \\ 0.08 \end{bmatrix}$$

$$= [15000 \times 0.1 + 20000 \times 0.08] = [1500 + 1600] = 3100$$

(iii) (a) ₹ 20000 in X, ₹ 15000 in Y
Explanation: Let ₹ x be invested in bond X and then ₹ (35000 - x) will be invested in bond Y.
Now, total amount of interest is given by

$$[x\ \ 35000 - x]\begin{bmatrix} 0.1 \\ 0.08 \end{bmatrix} = [0.1x + (35000 - x)\,0.08]$$

But, it is given that total amount of interest = ₹ 3200

$\therefore$ 0.1x + 2800 - 0.08x = 3200

$\Rightarrow$ 0.02 x = 400 $\Rightarrow$ x = 20000

Thus, ₹ 20000 invested in bond X and ₹ 35000 - ₹ 20000 = ₹ 15000 invested in bond Y.

(iv) (d) AB
Explanation: AB will give the total amount of interest received on both bonds.

(v) (c) ₹ 30000
Explanation: Let ₹ x invested in bond X, then we have

$x \times \frac{10}{100} = 500 \Rightarrow x = 5000$

Thus, the amount invested in bond X is ₹ 5000 and so investment in bond Y be ₹ (35000 - 5000) = ₹ 30000

➢ Questions

43. Ans. We have, $A^2 + I = kA$

$$\Rightarrow \begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = k\begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = k\begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 12 & -8 \\ -4 & 4 \end{bmatrix} = k\begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow -4\begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix} = k\begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix}$$

On comparing, we get k = -4

44. Ans. Given,

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

Now, $A^2 - 5A + 4I$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$+4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -1 & 2 \\ 9 & 2 & 5 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -3 \\ -1 & -3 & -10 \\ -5 & 4 & 2 \end{bmatrix}$$

Since,

$$A^2 - 5A + 4I + X = O \Rightarrow X = -(A^2 - 5A + 4I)$$

$$\therefore X = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 10 \\ 5 & -4 & -2 \end{bmatrix}$$

45. Ans.

$$A^2 = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} = \begin{bmatrix} 28 & 38 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix}$$

Now,

$$A^3 - 4A^2 - 3A + 11I = \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix} - 4\begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix}$$

$$-3\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix}28&37&26\\10&5&1\\35&42&34\end{bmatrix}-\begin{bmatrix}36&28&20\\4&16&4\\32&36&36\end{bmatrix}-\begin{bmatrix}3&9&6\\6&0&-3\\3&6&9\end{bmatrix}+\begin{bmatrix}11&0&0\\0&11&0\\0&0&11\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

Hence, $A^3-4A^2-3A+11I=O$

Now, $A^{-1}[A^3-4A^2-3A+11I]=A^{-1}O$

$\Rightarrow\ A^2-4A-3A^{-1}A+11A^{-1};=O$

$\Rightarrow\ A^2-4A-3I+11A^{-1}=O$

$$\Rightarrow A^{-1}=\frac{-A^2+4A+3I}{11}$$

$$\Rightarrow A^{-1}=\frac{1}{11}\left(\begin{bmatrix}-9&-7&-5\\-1&-4&-1\\-8&-9&-9\end{bmatrix}+\begin{bmatrix}4&12&8\\8&0&-4\\4&8&12\end{bmatrix}+\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}\right)$$

$$\Rightarrow A^{-1}=\frac{1}{11}\left(\begin{bmatrix}-2&5&3\\7&-1&-5\\-4&-1&6\end{bmatrix}\right)=\begin{bmatrix}-2/11&5/11&3/11\\7/11&-1/11&-5/11\\-4/11&-1/11&6/11\end{bmatrix}$$

46. Ans. Let, $A=\begin{bmatrix}0&6-5x\\x^2&x+3\end{bmatrix}$

A is symmetric, then A′ = A

$$\therefore\begin{bmatrix}0&x^2\\6-5x&x+3\end{bmatrix}=\begin{bmatrix}0&6-5x\\x^2&x+3\end{bmatrix}$$

On comparing both sides, we get

$\Rightarrow x^2=6-5x\Rightarrow x^2+5x-6=0$

$\Rightarrow (x+6)(x-1)=0\Rightarrow x=-6,1$

47. Ans. Given, $A=\begin{bmatrix}1&5\\6&7\end{bmatrix}$

(i) $A'=\begin{bmatrix}1&6\\5&7\end{bmatrix}$

$$A+A'=\begin{bmatrix}1&5\\6&7\end{bmatrix}+\begin{bmatrix}1&6\\5&7\end{bmatrix}=\begin{bmatrix}2&11\\11&14\end{bmatrix}$$

$$\therefore\ (A+A')'=\begin{bmatrix}2&11\\11&14\end{bmatrix}$$

$\Rightarrow\ (A+A')'=A+A'$

$\therefore\ (A+A')$ is a symmetric matrix.

(ii) $$A-A'=\begin{bmatrix}1&5\\6&7\end{bmatrix}-\begin{bmatrix}1&6\\5&7\end{bmatrix}=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$$

$$(A-A')'=\begin{bmatrix}0&1\\-1&0\end{bmatrix}=-\begin{bmatrix}0&-1\\1&0\end{bmatrix}=-(A-A')$$

$\Rightarrow$ $(A-A')$ is a skew symmetric matrix.

48. Soln.

Given $X=\begin{bmatrix}4&1\\-1&2\end{bmatrix}$

$$X^2=X.X=\begin{bmatrix}4&1\\-1&2\end{bmatrix}\begin{bmatrix}4&1\\-1&2\end{bmatrix}$$

$$=\begin{bmatrix}16-1&4+2\\-4-2&-1+4\end{bmatrix}$$

$$=\begin{bmatrix}15&6\\-6&3\end{bmatrix}$$

$$\therefore\ 6X-X^2=6\begin{bmatrix}4&1\\-1&2\end{bmatrix}-\begin{bmatrix}15&6\\-6&3\end{bmatrix}$$

$$=\begin{bmatrix}24&6\\-6&12\end{bmatrix}+\begin{bmatrix}-15&-6\\6&-3\end{bmatrix}$$

$$=\begin{bmatrix}24-15&6-6\\-6+6&12-3\end{bmatrix}=\begin{bmatrix}9&0\\0&9\end{bmatrix}$$

$$=9\begin{bmatrix}1&0\\0&1\end{bmatrix}=9I$$

49. Soln.

Given, $2x+Y=\begin{bmatrix}2&-3&1\\1&2&3\end{bmatrix}$

And $X-Y=\begin{bmatrix}0&3&5\\-2&-4&1\end{bmatrix}$

For finding the value of X, add 2X + Y and X – Y.

$$2X+Y+X-Y=\begin{bmatrix}2 & -3 & 1\\ 1 & 2 & 3\end{bmatrix}+\begin{bmatrix}0 & 3 & 5\\ -2 & -4 & 1\end{bmatrix}$$

$$3X=\begin{bmatrix}2 & 0 & 6\\ -1 & -2 & 4\end{bmatrix}$$

$$\therefore \quad X=\frac{1}{3}\begin{bmatrix}2 & 0 & 6\\ -1 & -2 & 4\end{bmatrix}$$

$$=\begin{bmatrix}2/3 & 0 & 2\\ -1/3 & -2/3 & 4/3\end{bmatrix}$$

On putting the value of X in (X − Y),

$$\begin{bmatrix}2/3 & 0 & 2\\ -1/3 & -2/3 & 4/3\end{bmatrix}-Y=\begin{bmatrix}0 & 3 & 5\\ -2 & -4 & 1\end{bmatrix}$$

$$Y=\begin{bmatrix}2/3 & 0 & 2\\ -1/3 & -2/3 & 4/3\end{bmatrix}-\begin{bmatrix}0 & 3 & 5\\ -2 & -4 & 1\end{bmatrix}$$

$$=\begin{bmatrix}2/3 & -3 & -3\\ 5/3 & 10/3 & 1/3\end{bmatrix}$$

Hence, $X=\begin{bmatrix}2/3 & 0 & 2\\ -1/3 & -2/3 & 4/3\end{bmatrix}$

$$Y=\begin{bmatrix}2/3 & -3 & -3\\ 5/3 & 10/3 & 1/3\end{bmatrix}$$

50. Soln.

Given,

$$\lambda\begin{bmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{bmatrix}+2\begin{bmatrix}2 & 5 & 6\\ 1 & 3 & 5\end{bmatrix}=\begin{bmatrix}7 & 19 & 27\\ 8 & 18 & 28\end{bmatrix}$$

Multiply by non-zero scalar λ to the corresponding matrix and then add to another matrix,

$$\begin{bmatrix}\lambda & 3\lambda & 0\\ 2\lambda & 4\lambda & 6\lambda\end{bmatrix}+\begin{bmatrix}1 & 0 & 12\\ 5 & 6 & 10\end{bmatrix}=\begin{bmatrix}7 & 19 & 27\\ 8 & 18 & 28\end{bmatrix}$$

$$\begin{bmatrix}\lambda & 3\lambda & 5\lambda\\ 2\lambda & 4\lambda & 6\lambda\end{bmatrix}=\begin{bmatrix}7 & 19 & 27\\ 8 & 18 & 28\end{bmatrix}-\begin{bmatrix}4 & 10 & 12\\ 2 & 6 & 10\end{bmatrix}$$

$$=\begin{bmatrix}7 & 19 & 27\\ 8 & 18 & 28\end{bmatrix}+\begin{bmatrix}-4 & -10 & -12\\ -2 & -6 & -10\end{bmatrix}$$

$$=\begin{bmatrix}7-4 & 19-10 & 27-12\\ 8-2 & 18-6 & 28-10\end{bmatrix}$$

$$=\begin{bmatrix}3 & 9 & 15\\ 6 & 12 & 18\end{bmatrix}$$

On equating the corresponding elements, we get

$$\lambda=3.$$

51. Soln.

Given, $A=\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}$

Now, $A^2=\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}$

$$A^2=\begin{bmatrix}4+0+1 & 0+0-1 & 2+0+0\\ 4+2+3 & 0+1-3 & 2+3+0\\ 2-2+0 & 0-1-0 & 1-3+0\end{bmatrix}$$

$$A^2=\begin{bmatrix}5 & -1 & 2\\ 9 & -2 & 5\\ 0 & -1 & -2\end{bmatrix}$$

And $5A=5\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}=\begin{bmatrix}10 & 0 & 5\\ 10 & 5 & 15\\ 5 & -5 & 0\end{bmatrix}$

$$\therefore A^2-5A=\begin{bmatrix}2 & -1 & 2\\ 9 & -2 & 5\\ 0 & -1 & -2\end{bmatrix}-\begin{bmatrix}10 & 0 & 5\\ 10 & 5 & 15\\ 5 & -5 & 0\end{bmatrix}$$

$$A^2-5A=\begin{bmatrix}-8 & -1 & -3\\ -1 & -7 & -10\\ -5 & 4 & -2\end{bmatrix}$$

52. Soln.

We have, $\begin{bmatrix}2 & -1\\ 1 & 0\\ -3 & 4\end{bmatrix}_{3\times 2} A=\begin{bmatrix}-1 & -8\\ 1 & -2\\ 9 & 22\end{bmatrix}_{3\times 2}$

∴ A is of order 2 x 2

Let $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

Then, $\begin{bmatrix}2 & -1\\ 1 & 0\\ -3 & 4\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}-1 & -8\\ 1 & -2\\ 9 & 22\end{bmatrix}$

$$\begin{bmatrix}2a-c & 2b-d\\ a & b\\ -3a+4c & -3b+4d\end{bmatrix}=\begin{bmatrix}-1 & -8\\ 1 & -2\\ 9 & 22\end{bmatrix}$$

By equality of matrices, on comparing, we get

$$2a-c=-1$$
$$2b-d=-8$$
$$a=1$$
$$b=-2$$
$$-3a+4c=9$$
$$-3b+4d=22$$

On solving the equations, we get

$a=1,\quad b=-2,\quad c=3,\quad d=4$

Hence, $A=\begin{bmatrix}1 & -2\\ 3 & 4\end{bmatrix}$ Ans.

53. Soln. Given, B = $\begin{bmatrix}-2 & 2 & 0\\ 3 & 1 & 4\end{bmatrix}$ *and* $C=\begin{bmatrix}2 & 0 & -2\\ 7 & 1 & 6\end{bmatrix}$

And $2A-3B+5C=0$

$$\Rightarrow \quad 2A=-5C+3B$$
$$\Rightarrow \quad A=\frac{1}{2}[3B-5C]$$
$$\Rightarrow \quad A=\frac{1}{2}\left(3\begin{bmatrix}-2 & 2 & 0\\ 3 & 1 & 4\end{bmatrix}-5\begin{bmatrix}2 & 0 & -2\\ 7 & 1 & 6\end{bmatrix}\right)$$
$$\Rightarrow \quad A=\frac{1}{2}\left(\begin{bmatrix}-6 & 6 & 0\\ 9 & 3 & 12\end{bmatrix}-\begin{bmatrix}10 & 0 & -10\\ 35 & 5 & 30\end{bmatrix}\right)$$
$$\Rightarrow \quad A=\frac{1}{2}\begin{bmatrix}-16 & 6 & 10\\ -26 & -2 & -18\end{bmatrix}$$
$$\Rightarrow \quad A=\begin{bmatrix}-8 & 3 & 5\\ -13 & -1 & -9\end{bmatrix} \quad \text{Ans.}$$

54. Soln. Given,

$$(A-I)^3+(A+I)^3-7A$$
$$=A^3-I^3-3A^2I+3AI^2+A^3+I^3+3A^2I+3AI^2-7A$$
$$=2A^3+6AI^2-7A$$
$$=2A.A^2+6AI^2-7A$$
$$=2AI+6AI-7A$$
$$=8A-7A$$
$$=A$$

55. Soln. Since, $A^2=A$,

$$7A-(I+A)^3=7A-I^3-3A^2I-3AI^2-A^3$$
$$=7A-I-3A-3A-A^2A$$
$$=7A-I-3A-3A-A.A$$
$$=7A-I-3A-3A-A$$
$$=7A-I-7A$$
$$=-I$$

56. Soln.

$$\begin{bmatrix}a+4 & 3b\\ 8 & -6\end{bmatrix}=\begin{bmatrix}2a+2 & b+2\\ 8 & a-8b\end{bmatrix}$$

By equating, $a+4=2a+2$ *or* $a=2$

$3b=b+2$ *or* $b=1$

$$\therefore \quad a-2b=2-2(1)$$
$$=2-2$$
$$=0$$

57. Soln. Given $A^2=\lambda A$, where $A=\begin{bmatrix}3 & -3\\ -3 & 3\end{bmatrix}$

$$A^2=\begin{bmatrix}3 & -3\\ -3 & 3\end{bmatrix}\begin{bmatrix}3 & -3\\ -3 & 3\end{bmatrix}$$
$$=\begin{bmatrix}9+9 & -9-9\\ -9-9 & 9+9\end{bmatrix}=\begin{bmatrix}18 & -18\\ -18 & 18\end{bmatrix}$$

Since $A^2=\lambda A$ *or* $\begin{bmatrix}18 & -18\\ -18 & 18\end{bmatrix}=\begin{bmatrix}3\lambda & -3\lambda\\ -3\lambda & 3\lambda\end{bmatrix}$

Or $18=3\lambda$ *or* $\lambda=6$

58. Soln. We have, $X+Y=\begin{bmatrix}2 & 3\\ 5 & 1\end{bmatrix}, X-Y=\begin{bmatrix}6 & 5\\ 7 & 3\end{bmatrix}$

$$(X+Y)+(X-Y)=\begin{bmatrix}2 & 3\\ 5 & 1\end{bmatrix}+\begin{bmatrix}6 & 5\\ 7 & 3\end{bmatrix}$$
$$2X=\begin{bmatrix}8 & 8\\ 12 & 4\end{bmatrix}$$

Or $X=\begin{bmatrix}4 & 4\\ 6 & 2\end{bmatrix}$

$$\therefore \quad X=\begin{bmatrix}4 & 4\\ 6 & 2\end{bmatrix}$$

And $Y=\begin{bmatrix}2 & 3\\ 5 & 1\end{bmatrix}-\begin{bmatrix}4 & 4\\ 6 & 2\end{bmatrix}$

$$=\begin{bmatrix}-2 & -1\\ -1 & -1\end{bmatrix}$$

59. Soln. Getting $A^2 = A.A$

$$=\begin{bmatrix}2&0&1\\2&1&3\\1&-1&0\end{bmatrix}\begin{bmatrix}2&0&1\\2&1&3\\1&-1&0\end{bmatrix}=\begin{bmatrix}5&-1&2\\9&-2&5\\0&-1&-2\end{bmatrix}$$

$$\therefore A^2-5A+4I=\begin{bmatrix}5&-1&2\\9&-2&5\\0&-1&-2\end{bmatrix}-5\begin{bmatrix}2&0&1\\2&1&3\\1&-1&0\end{bmatrix}+4\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}5&-1&2\\9&-2&5\\0&-1&-2\end{bmatrix}+\begin{bmatrix}-10&0&-5\\-10&-5&-15\\-5&5&0\end{bmatrix}+\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}$$

$$=\begin{bmatrix}-1&-1&-3\\-1&-3&-10\\-5&4&2\end{bmatrix}$$

Getting $X=-(A^2-5A+4I)$

i.e., $X=\begin{bmatrix}1&1&3\\1&3&10\\5&-4&-2\end{bmatrix}$

60. Soln.

We have, $A=\begin{bmatrix}0&a&b\\-a&0&c\\-b&-c&0\end{bmatrix}$

$$A'=\begin{bmatrix}0&-a&-b\\a&0&-c\\b&c&0\end{bmatrix}$$

$$A+A'=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

Or $\frac{1}{2}(A+A')=\frac{1}{2}\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=0$

And $A-A'=\begin{bmatrix}0&2a&2b\\-2a&0&2c\\-2b&-2c&0\end{bmatrix}$

Or $\frac{1}{2}(A-A')=\frac{1}{2}\begin{bmatrix}0&2a&2b\\-2a&0&2c\\-2b&-2c&0\end{bmatrix}$

Or $\frac{1}{2}(A+A')=\begin{bmatrix}0&a&b\\-a&0&c\\-b&-c&0\end{bmatrix}$

61. Soln. We know that a square matrix A can be written as

$$A=\frac{1}{2}(A+A^T)+\frac{1}{2}(A-A^T)$$

Out of which $\frac{1}{2}(A+A^T)$ is symmetric and $\frac{1}{2}(A-A^T)$ is skew symmetric matrix.

$\therefore$ For the given matrix

$$A=\begin{bmatrix}2&4&-6\\7&3&5\\1&-2&4\end{bmatrix} \text{ and } A^T=\begin{bmatrix}2&7&1\\4&3&-2\\-6&5&4\end{bmatrix}$$

$$\therefore A+A^T=\begin{bmatrix}4&11&-5\\11&6&3\\-5&3&8\end{bmatrix} \text{ and } A-A^T=\begin{bmatrix}0&-3&-7\\3&0&7\\7&-7&0\end{bmatrix}$$

Hence, $A=\frac{1}{2}(A+A^T)+\frac{1}{2}(A-A^T)$

$$=\begin{bmatrix}2&11/2&-5/2\\11/2&3&3/2\\-5/2&3/2&4\end{bmatrix}+\begin{bmatrix}0&-3/2&-7/2\\3/2&0&7/2\\7/2&-7/2&0\end{bmatrix}$$

$$=\begin{bmatrix}2&4&-6\\7&3&5\\1&-2&4\end{bmatrix}=A$$

62. Soln. We know that

$$A=\frac{1}{2}(A+A')+\frac{1}{2}(A-A')$$

Here, $\frac{1}{2}(A+A')$ is symmetric matrix and

$\frac{1}{2}(A-A')$ is skew symmetric matrix.

Now, $A=\begin{bmatrix}3 & -2 & -4\\ 3 & -2 & -5\\ -1 & 1 & 2\end{bmatrix} \Rightarrow A'=\begin{bmatrix}3 & 3 & -1\\ -2 & -2 & 1\\ -4 & -5 & 2\end{bmatrix}$

$$\therefore \frac{1}{2}(A+A')=\frac{1}{2}\begin{bmatrix}3+3 & -2+3 & -4-1\\ 3-2 & -2-2 & -5+1\\ -1-4 & 1-5 & 2+2\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}6 & 1 & -5\\ 1 & -4 & -4\\ -5 & -4 & 4\end{bmatrix} \text{ which is symmetric}$$

$$\frac{1}{2}(A-A')=\frac{1}{2}\begin{bmatrix}3-3 & -2-3 & -4+1\\ 3+2 & -2+2 & -5-1\\ -1+4 & 1+5 & 2-2\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}0 & -5 & -3\\ 5 & 0 & -6\\ 3 & 6 & 0\end{bmatrix} \text{ which is symmetric.}$$

$$\therefore A=\frac{1}{2}\begin{bmatrix}6 & 1 & -5\\ 1 & -4 & -4\\ -5 & -4 & 4\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0 & -5 & -3\\ 5 & 0 & -6\\ 3 & 6 & 0\end{bmatrix}$$

$$\Rightarrow A=\begin{bmatrix}3 & 1/2 & -5/2\\ 1/2 & -2 & -2\\ -5/2 & -2 & 2\end{bmatrix}+\begin{bmatrix}0 & -5/2 & -3/2\\ 5/2 & 0 & -3\\ 3/2 & 3 & 0\end{bmatrix}$$

$$=\begin{bmatrix}3 & -2 & -4\\ 3 & -2 & -5\\ -1 & 1 & 2\end{bmatrix}$$

63. Soln. Case I: Let A be a symmetric matrix. Then $A^T = A$.

Now, $(B^TAB)^T = B^TA^T(B^T)^T$ [By reversal law]

$= B^TA^TB$ $[\because (B^T)^T = B]$

Or $(B^TAB)^T = B^TAB$ $[\because A^T = A]$

$\therefore B^TAB$ is a symmetric matrix.

Case II: Let A be a skew – symmetric matrix. Then, $A^T = -A$.

Now, $(B^TAB)^T = B^TA^T(B^T)^T$ [By reversal law]

Or $(B^TAB)^T = B^TA^TB$ $[\because (B^T)^T = B]$

Or $(B^TAB)^T = B^T(-A)B$ $[\because A^T = -A]$

Or $(B^TAB)^T = -B^TAB$

$\therefore B^T AB$ is a skew – symmetric matrix.

64. Soln. We shall prove the result by using principle of mathematical induction.

Let $P(n): A^n = \begin{bmatrix}3^{n-1} & 3^{n-1} & 3^{n-1}\\ 3^{n-1} & 3^{n-1} & 3^{n-1}\\ 3^{n-1} & 3^{n-1} & 3^{n-1}\end{bmatrix}$

Now, $P(1): A^1 = \begin{bmatrix}3^0 & 3^0 & 3^0\\ 3^0 & 3^0 & 3^0\\ 3^0 & 3^0 & 3^0\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$

The result is true for n = 1.

Let the result be true for n = k.

So, $A^k = \begin{bmatrix}3^{k-1} & 3^{k-1} & 3^{k-1}\\ 3^{k-1} & 3^{k-1} & 3^{k-1}\\ 3^{k-1} & 3^{k-1} & 3^{k-1}\end{bmatrix}$

Now, we prove that P(k + 1) is true.

Now $A^{k+1} = A.A^k$

$$=\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}3^{k-1} & 3^{k-1} & 3^{k-1}\\ 3^{k-1} & 3^{k-1} & 3^{k-1}\\ 3^{k-1} & 3^{k-1} & 3^{k-1}\end{bmatrix}$$

$$=\begin{bmatrix}3.3^{k-1} & 3.3^{k-1} & 3.3^{k-1}\\ 3.3^{k-1} & 3.3^{k-1} & 3.3^{k-1}\\ 3.3^{k-1} & 3.3^{k-1} & 3.3^{k-1}\end{bmatrix}$$

$$=\begin{bmatrix}3^k & 3^k & 3^k\\ 3^k & 3^k & 3^k\\ 3^k & 3^k & 3^k\end{bmatrix}$$

$= A^{k+1}$

Hence, it is true n = k + 1.

Hence, by principle of mathematical induction P(n) is true for all $n \in N$.

65. Soln. We have, $\begin{bmatrix}x^2-4x & x^2\\ x^2 & x^3\end{bmatrix} = \begin{bmatrix}-3 & 1\\ -x+2 & 1\end{bmatrix}$

Equating the corresponding elements of two matrices, we get

$x^2-4x=-3 \Rightarrow x^2-4x+3=0 \Rightarrow (x-1)(x-3)=0$

$\Rightarrow\ x=1,3$

$x^2=1 \Rightarrow x=\pm 1$

$x^2=-x+2 \Rightarrow x^2+x-2=0$

$\Rightarrow (x-1)(x+2)=0 \Rightarrow x=1,-2$

$x^3=1 \Rightarrow x^3-1=0 \Rightarrow (x-1)(x^2+x+1)=0$

$\Rightarrow\ x=1,\ \omega,\omega^2,$

Where, $\omega=\dfrac{-1+\sqrt{3}i}{2},\ \omega^2=\dfrac{-1-\sqrt{3}i}{2}$

Since, common value of x is 1.

$\therefore\quad x=1$

66. Soln. We have,

$$A=\begin{bmatrix} p & q \\ r & s \end{bmatrix} \Rightarrow A'=\begin{bmatrix} p & r \\ q & s \end{bmatrix}$$

$$\therefore\ A+A'=\begin{bmatrix} p & q \\ r & s \end{bmatrix}+\begin{bmatrix} p & r \\ q & s \end{bmatrix}=\begin{bmatrix} 2p & q+r \\ q+r & 2s \end{bmatrix}$$

$$\Rightarrow \frac{1}{2}(A+A')=\frac{1}{2}\begin{bmatrix} 2p & q+r \\ q+r & 2s \end{bmatrix}=\begin{bmatrix} p & \frac{q+r}{2} \\ \frac{q+r}{2} & s \end{bmatrix}$$

$$\text{Also, } A-A'=\begin{bmatrix} p & q \\ r & s \end{bmatrix}-\begin{bmatrix} p & r \\ q & s \end{bmatrix}=\begin{bmatrix} 0 & q-r \\ r-q & 0 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2}(A-A')=\frac{1}{2}\begin{bmatrix} 0 & q-r \\ r-q & 0 \end{bmatrix}=\begin{bmatrix} 0 & \frac{q-r}{2} \\ \frac{r-q}{2} & 0 \end{bmatrix}$$

$$\therefore\ A=\frac{1}{2}(A+A')+\frac{1}{2}(A-A')$$

$$=\begin{bmatrix} p & \frac{q+r}{2} \\ \frac{q+r}{2} & s \end{bmatrix}+\begin{bmatrix} 0 & \frac{q-r}{2} \\ \frac{r-q}{2} & 0 \end{bmatrix}$$

67. Soln. We have,

$$A=\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} and\ B=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\text{Now, } 4A+2B=4\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}+2\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} 8 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 12 \end{bmatrix}+\begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}=\begin{bmatrix} 14 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

$\therefore\ 4A+2B=diag[14\ \ -4\ \ 10]$

68. Sol. We observe that there are 3 rows and 4 columns in matrix A.

$\therefore$ It is of order 3 x 4.

Here, $a_{32}=15,\ a_{23}=9,\ a_{24}=6$

$\therefore\ a_{23}+a_{24}=9+6=15=a_{32}$, which is true.

69. Soln. We have given,

$$A=\begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} and\ A'=\begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}$$

Since, A' = A^{-1}

$\therefore\ AA'=AA^{-1}$ [Multiplying by A on both sides]

$\Rightarrow\ AA'=I$

$$\Rightarrow \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}\begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4y^2+z^2 & 2y^2-z^2 & -2y^2+z^2 \\ 2y^2-z^2 & x^2+y^2+z^2 & x^2-y^2-z^2 \\ -2y^2+z^2 & x^2-y^2-z^2 & x^2+y^2+z^2 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By equality of two matrices, we get

$\Rightarrow\ 2y^2-z^2=0 \Rightarrow 2y^2=z^2$ ....(i)

And $4y^2+z^2=1$

$\Rightarrow 2.z^2+z^2=1$ [Using (i)]

$\Rightarrow\ z=\pm\dfrac{1}{\sqrt{3}} \because\ y^2=\dfrac{z^2}{2} \Rightarrow y=\pm\dfrac{1}{\sqrt{6}}$

Also, $x^2+y^2+z^2=1$

$\Rightarrow x^2=1-y^2-z^2=1-\dfrac{1}{6}-\dfrac{1}{3}=\dfrac{1}{2}$

$\Rightarrow x=\pm\dfrac{1}{\sqrt{2}} \therefore\ x=\pm\dfrac{1}{\sqrt{2}},\ y=\pm\dfrac{1}{\sqrt{6}}\ and\ z=\pm\dfrac{1}{\sqrt{3}}$

70. Soln. Given, A = $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+4 & 0+0+0 & 2+0+6 \\ 0+0+2 & 0+4+0 & 0+2+3 \\ 2+2+6 & 0+0+0 & 4+0+9 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$A^3 = A.A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 5+0+16 & 0+0+0 & 8+0+26 \\ 0+4+8 & 0+8+0 & 0+10+13 \\ 10+0+24 & 0+0+0 & 16+0+39 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

Since, $A^3 - 6A^2 + 7A + kI_3 = O$

$$\Rightarrow \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$+k\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix}$$

$$+\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2+k & 0 & 0 \\ 0 & -2+k & 0 \\ 0 & 0 & -2+k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

On equating the corresponding elements, we get
$-2+k = 0 \Rightarrow k = 2$

71. Soln. We have,

$$A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\Rightarrow A' = \begin{bmatrix} 3 & 4 \\ \sqrt{3} & 2 \\ 2 & 0 \end{bmatrix} \text{ and } B' = \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix}$$

(i) $(A')' = \begin{bmatrix} 3 & 4 \\ \sqrt{3} & 2 \\ 2 & 0 \end{bmatrix}' = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix} = A$

(ii) $A + B = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 3+2 & \sqrt{3}-1 & 2+2 \\ 4+1 & 2+2 & 0+4 \end{bmatrix} = \begin{bmatrix} 5 & \sqrt{3}-1 & 4 \\ 5 & 4 & 4 \end{bmatrix}$$

$$\Rightarrow (A+B)' = \begin{bmatrix} 5 & \sqrt{3}-1 & 4 \\ 5 & 4 & 4 \end{bmatrix}' = \begin{bmatrix} 5 & 5 \\ \sqrt{3}-1 & 4 \\ 4 & 4 \end{bmatrix}$$

Also,

$$A' + B' = \begin{bmatrix} 3 & 4 \\ \sqrt{3} & 2 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ \sqrt{3}-1 & 4 \\ 4 & 4 \end{bmatrix}$$

Thus, (A + B)′ = A′ + B′

(iii) $kB = k\begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2k & -k & 2k \\ k & 2k & 4k \end{bmatrix}$

And

$$(kB)' = \begin{bmatrix} 2k & -k & 2k \\ k & 2k & 4k \end{bmatrix}' = \begin{bmatrix} 2k & k \\ -k & 2k \\ 2k & 4k \end{bmatrix} = k\begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix} = kB'$$

Thus (kB)′ = kB′.

72. Soln. The matrix $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$ is skew – symmetric.

$$\therefore A' = -A \Rightarrow \begin{bmatrix} 0 & -1 & x \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -x & 3 & 0 \end{bmatrix} \Rightarrow x = 2$$

For Solutions

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SURE SHOT QUESTIONS

Chapter – 04 (Solution)

Determinants

➢ MCQ (1 mark)

1. Soln. (c): $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix} \Rightarrow 2x^2 - 40 = 18 + 14$

$\Rightarrow x^2 = 36 \Rightarrow x = \pm 6$

2. Soln. (c): Let $\Delta = \begin{vmatrix} a-b & b+c & a \\ b-c & c+a & b \\ c-a & a+b & c \end{vmatrix}$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} -b & a+b+c & a \\ -c & a+b+c & b \\ -a & a+b+c & c \end{vmatrix}$$

Taking (a + b + c) common from C_2, we get

$$\Delta = (a+b+c)\begin{vmatrix} -b & 1 & a \\ -c & 1 & b \\ -a & 1 & c \end{vmatrix}$$

Applying, $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$, we get

$$\Delta = (a+b+c)\begin{vmatrix} a-b & 0 & a-c \\ a-c & 0 & b-c \\ -a & 1 & c \end{vmatrix}$$

Expanding along C_2, we get

$$\Delta = (a+b+c)(-1)[(a-b)(b-c)-(a-c)^2]$$

$$= (a+b+c)(a^2+b^2+c^2-ab-bc-ac)$$

$$= a^3+b^3+c^3-3abc$$

3. Soln. (b): Area of triangle = $\frac{1}{2}\begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix} = \pm 9$

$\Rightarrow -k(-3-3) = \pm 18 \Rightarrow 6k = \pm 18 \Rightarrow k = \pm 3$

4. Soln. (d): Let $\Delta = \begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$

$$= \begin{vmatrix} b(b-a) & b-c & c(b-a) \\ a(b-a) & a-b & b(b-a) \\ c(b-a) & c-a & a(b-a) \end{vmatrix}$$

Taking (b – a) common from C_1 and C_3, we get

$$\Delta = (b-a)^2\begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$, we get

$$\Delta = (b-a)^2\begin{vmatrix} c & b-c & c \\ b & a-b & b \\ a & c-a & a \end{vmatrix}$$

Since, C_1 *and* C_3 are identical $\Rightarrow \Delta = 0$

5. Soln. (c): $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} \sin x + 2\cos x & \cos x & \cos x \\ \sin x + 2\cos x & \sin x & \cos x \\ 2\cos x + \sin x & \cos x & \sin x \end{vmatrix} = 0$$

Taking $(\sin x + 2\cos x)$ common from C_1, WE GET

$$(\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$(\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x - \cos x & 0 \\ 0 & 0 & \sin x - \cos x \end{vmatrix} = 0$$

$$\Rightarrow (\sin x+2\cos x)(\sin x-\cos x)^2=0$$

$$\Rightarrow \sin x+2\cos x=0 \text{ or } (\sin x-\cos x)^2=0$$

$$\Rightarrow \tan x=-2 \text{ or } \tan x=1$$

$$\Rightarrow x=-\tan^{-1}(2) \text{ or } x=\frac{\pi}{4}$$

$$\because\ -\tan^{-1}(2)\notin\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$$

So, $x=\dfrac{\pi}{4}$

6. Soln. (a): Let $\Delta=\begin{vmatrix}-1 & \cos C & \cos B\\ \cos C & -1 & \cos A\\ \cos B & \cos A & -1\end{vmatrix}$

Applying $C_2\to C_2+(\cos C)C_1$ and $C_3\to C_3+(\cos B)C_1$, we get

$$\Delta=\begin{vmatrix}-1 & 0 & 0\\ \cos C & -1+\cos^2 C & \cos A+\cos C\cos B\\ \cos B & \cos A+\cos B\cos C & -1+\cos^2 B\end{vmatrix}$$

$$=\begin{vmatrix}-1 & 0 & 0\\ \cos C & -\sin^2 C & \cos A+\cos C\cos B\\ \cos B & \cos A+\cos B\cos C & -\sin^2 B\end{vmatrix}$$

Expanding along R_1, we get

$$\Delta=-1[\sin^2C\sin^2B-(\cos A+\cos C\cos B)(\cos A+\cos B\cos C)]$$

$$=-[\sin^2C\sin^2B-\cos^2A-\cos A\cos B\cos C-\cos A\cos B\cos C-\cos^2B\cos^2C]$$

$$=-[(1-\cos^2C)(1-\cos^2B)-\cos^2A-2\cos A\cos B\cos C-\cos^2B\cos^2C]$$

$$=-[1-\cos^2C-\cos^2B+\cos^2B\cos^2C-\cos^2A-2\cos A\cos B\cos C-\cos^2B\cos^2C]$$

$$=-[1-\cos^2C-\cos^2B-\cos^2A-2\cos A\cos B\cos C]$$

$$=-1+\cos^2A+\cos^2B+\cos^2C+2\cos A\cos B\cos C$$

Now, in a triangle

$$\cos^2A+\cos^2B+\cos^2C+2\cos A\cos B\cos C=1$$

$$\Rightarrow\ \Delta=-1+1=0$$

7. Soln. (a): $f(t)=\begin{vmatrix}\cos t & t & 1\\ 2\sin t & t & 2t\\ \sin t & t & t\end{vmatrix}$

Applying $R_1\to R_1-R_3$ and $R_2\to R_2-R_3$, we get

$$f(t)=\begin{vmatrix}\cos t-\sin t & 0 & 1-t\\ \sin t & 0 & t\\ \sin t & t & t\end{vmatrix}$$

Expanding along C_2, we get

$$f(t)=-t(t\cos t-t\sin t-\sin t+t\sin t)$$

$$=-t(t\cos t-\sin t)$$

Now, $\displaystyle\lim_{t\to0}\frac{f(t)}{t^2}=\lim_{t\to0}\frac{-t(t\cos t-\sin t)}{t^2}$

$$=\lim_{t\to0}\left(-\frac{(t\cos t-\sin t)}{t}\right)=\lim_{t\to0}\left(-\cos t+\frac{\sin t}{t}\right)$$

$$=-\lim_{t\to0}\cos t+\lim_{t\to0}\frac{\sin t}{t}=-1+1=0$$

8. Soln. (a): $\Delta=\begin{vmatrix}1 & 1 & 1\\ 1 & 1+\sin\theta & 1\\ 1+\cos\theta & 1 & 1\end{vmatrix}$

Applying $R_1\to R_1-R_3$ and $R_2\to R_2-R_3$, we get

$$\Delta=\begin{vmatrix}-\cos\theta & 0 & 0\\ -\cos\theta & \sin\theta & 0\\ 1+\cos\theta & 1 & 1\end{vmatrix}$$

Expanding along C_3, we get

$$\Delta=1(-\cos\theta\sin\theta-0)=-\cos\theta\sin\theta=\frac{-1}{2}\sin2\theta$$

We know, $-1\le\sin2\theta\le1$

$$\Rightarrow -\frac{1}{2}\le\frac{1}{2}\sin\theta\le\frac{1}{2}\Rightarrow\frac{1}{2}\ge-\frac{1}{2}\sin2\theta\ge\frac{-1}{2}$$

i.e., $-\dfrac{1}{2}\le-\dfrac{1}{2}\sin2\theta\le\dfrac{1}{2}\Rightarrow\dfrac{-1}{2}\le\Delta\le\dfrac{1}{2}$

Hence, maximum value of Δ is $\dfrac{1}{2}$.

9. Soln. (c): $f(x)=\begin{vmatrix}0 & x-a & x-b\\ x+a & 0 & x-c\\ x+b & x+c & 0\end{vmatrix}$

Expanding along R_1, we get

$$f(x)=0-(x-a)[0-(x-c)(x+b)+(x-b)[(x+a)(x+c)-0]$$

$$\Rightarrow f(x)=(x-a)(x-c)(x+b)+(x-b)(x+a)(x+c)$$

Now, $f(0)=(0-a)(0-c)(0+b)+(0-b)(0+a)(0+c)$

$$=(-a)(-c)(b)+(-b)(a)(c)=abc-abc=0$$

10. Soln. (d): A^{-1} exists if $|A|\ne0$

i.e., $\begin{vmatrix}2 & \lambda & -3\\ 0 & 2 & 5\\ 1 & 1 & 3\end{vmatrix}\ne0$

$$\Rightarrow 2(6-5)+1(5\lambda+6)\ne0\ \text{(Expanding along } C_1\text{)}$$

$$\Rightarrow 2+5\lambda+6\ne0\Rightarrow5\lambda\ne-8\ i.e.,\ \lambda\ne\frac{-8}{5}$$

11. Soln. (d): $(A+B)^{-1} \neq B^{-1}+A^{-1}$

e.g., $A=\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}$, $B=\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}$

Then, $A+B=\begin{bmatrix}9 & 15\\ 9 & 14\end{bmatrix}$

$\therefore |A+B| = 9\times14-9\times15=-9\neq 0$

So, (A + B) is invertible.

$$adj(A+B)=\begin{bmatrix}14 & -15\\ -9 & 9\end{bmatrix}$$

$$\Rightarrow (A+B)^{-1}=\frac{-1}{9}\begin{bmatrix}14 & -15\\ -9 & 9\end{bmatrix}=\begin{bmatrix}\frac{-14}{9} & \frac{15}{9}\\ 1 & -1\end{bmatrix}$$

Now, |A| = 3 x 5 -2 x 7 = 15 – 14 = $1\neq 0$

And $|B| = 6\times9-7\times8=54-56=-2\neq0$

So, A and B both are invertible.

$$A^{-1}=\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix} \text{ and } B^{-1}=\frac{-1}{2}\begin{bmatrix}9 & -8\\ -7 & 6\end{bmatrix}$$

$$\Rightarrow B^{-1}=\begin{bmatrix}\frac{-9}{2} & 4\\ \frac{7}{2} & -3\end{bmatrix}$$

$$\text{Now, } B^{-1}+A^{-1}=\begin{bmatrix}-\frac{9}{2} & 4\\ \frac{7}{2} & -3\end{bmatrix}+\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{2} & -3\\ \frac{3}{2} & 0\end{bmatrix}\neq\begin{bmatrix}\frac{-14}{9} & \frac{15}{9}\\ 1 & -1\end{bmatrix}=(A+B)^{-1}$$

Hence, $(A+B)^{-1}\neq B^{-1}+A^{-1}$

12. Soln. (d): $\begin{vmatrix}1+x & 1 & 1\\ 1 & 1+y & 1\\ 1 & 1 & 1+z\end{vmatrix}=0$

Taking x, y and z common from C_1, C_2 and C_3 respectively, we get

$$xyz\begin{vmatrix}1+\frac{1}{x} & \frac{1}{y} & \frac{1}{z}\\ \frac{1}{x} & 1+\frac{1}{y} & \frac{1}{z}\\ \frac{1}{x} & \frac{1}{y} & 1+\frac{1}{z}\end{vmatrix}=0$$

Applying $C_1\to C_1+C_2+C_3$, we get

$$xyz\begin{vmatrix}1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} & \frac{1}{y} & \frac{1}{z}\\ 1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} & 1+\frac{1}{y} & \frac{1}{z}\\ 1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} & \frac{1}{y} & 1+\frac{1}{z}\end{vmatrix}=0$$

Taking $\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$ common from C_1, we get

$$(xyz)\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\begin{vmatrix}1 & \frac{1}{y} & \frac{1}{z}\\ 1 & 1+\frac{1}{y} & \frac{1}{z}\\ 1 & \frac{1}{y} & 1+\frac{1}{z}\end{vmatrix}=0$$

Applying $R_1\to R_1-R_3$, $R_2\to R_2-R_3$, we get

$$(xyz)\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\begin{vmatrix}0 & 0 & -1\\ 0 & 1 & -1\\ 1 & \frac{1}{y} & 1+\frac{1}{z}\end{vmatrix}=0$$

Expanding along C_1, we get

$$(xyz)\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right).1(0+1)=0$$

$$\Rightarrow (xyz)\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=0$$

Since, x, y and z are different from zero.

So, $xyz\neq0$

$$\Rightarrow 1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$$

$$\Rightarrow \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=-1$$

$$\Rightarrow x^{-1}+y^{-1}+z^{-1}=-1$$

13. Soln. (b): Let $\Delta=\begin{vmatrix}x & x+y & x+2y\\ x+2y & x & x+y\\ x+y & x+2y & x\end{vmatrix}$

Applying $C_1\to C_1+C_2+C_3$, we get

$$\Delta=\begin{vmatrix}3x+3y & x+y & x+2y\\ 3x+3y & x & x+y\\ 3x+3y & x+2y & x\end{vmatrix}$$

Taking $(3x+3y)$ common from C_1, we get

$$\Delta=(3x+3y)\begin{vmatrix}1 & x+y & x+2y\\ 1 & x & x+y\\ 1 & x+2y & x\end{vmatrix}$$

Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$, we get

$$\Delta=(3x+3y)\begin{vmatrix}0 & -y & 2y\\ 0 & -2y & y\\ 1 & x+2y & x\end{vmatrix}$$

Expanding along C_1, we get

$$\Delta=(3x+3y)1(-y^2 4y^2)=3(x+y)(3y^2)=9y^2(x+y)$$

14. Soln. (c): $\Delta=\begin{vmatrix}1 & -2 & 5\\ 2 & a & -1\\ 0 & 4 & 2a\end{vmatrix}=86$

Applying $R_2 \to R_2 - 2R_1$, we get

$$\begin{vmatrix}1 & -2 & 5\\ 0 & a+4 & -11\\ 0 & 4 & 2a\end{vmatrix}=86$$

Expanding along C_1, we get

$1[(a+4)2a-(4)(-11)]=86$

$\Rightarrow 2a^2+8a+44=86$

$\Rightarrow a^2+4a-21=0$

$\Rightarrow (a+7)(a-3)=0 \Rightarrow a=3,-7$

Now, sum of values of a is 3 + (-7) = 3 – 7 = -4

15. Ans. (a): Let $|A|=\begin{vmatrix}2 & 7 & 1\\ 1 & 1 & 1\\ 10 & 8 & 1\end{vmatrix}$

Expanding along R_1, we get

$|A|=2(1-8)-7(1-10)+1(8-10)$

$=2(-7)-7(-9)+1(-2)$

$=-14+63-2=47$

16. Ans. (d):

$$\begin{vmatrix}\alpha & 3 & 4\\ 1 & 2 & 1\\ 1 & 4 & 1\end{vmatrix}=0 \Rightarrow \alpha(2-4)-3(1-1)+4(4-2)=0$$

$\Rightarrow -2\alpha+8=0 \Rightarrow 2\alpha=8 \Rightarrow \alpha=4$

17. Ans. (d): Given, $A=\begin{bmatrix}\alpha & 2\\ 2 & \alpha\end{bmatrix}$

..........(i)

$|A^3|=27$

$\Rightarrow |A|^3=27\ [\because |A^n|=|A|^n] \Rightarrow |A|=3$ ....(ii)

From (i) and (ii), we get

$\Rightarrow \alpha^2-4=3 \Rightarrow \alpha^2=7 \Rightarrow \alpha=\pm\sqrt{7}$

18. Ans. (d): We have, $\begin{vmatrix}5 & 3 & -1\\ -7 & x & -3\\ 9 & 6 & -2\end{vmatrix}=0$

$\Rightarrow 5(-2x+18)-3(14+27)-1(-42-9x)=0$

$\Rightarrow -x+9=0$

$\Rightarrow x=9$

19. Ans. (d): Let $D=\begin{vmatrix}y+k & y & y\\ y & y+k & y\\ y & y & y+k\end{vmatrix}$

$=(y+k)((y+k)^2-y^2)-y(y^2+ky-y^2)+y(y^2-y^2-yk)$

$=k^3+3k^2y=k^2(k+3y)$

20. Ans. (d): Given, $\begin{vmatrix}1! & 2! & 3!\\ 2\cdot 2! & 3\cdot 3! & 4\cdot 4!\\ 3! & 4! & 5!\end{vmatrix}=\begin{vmatrix}1 & 2 & 6\\ 4 & 18 & 96\\ 6 & 24 & 120\end{vmatrix}$

= 1(2160 - 2304) – 2(480 – 576) + 6(96 – 108)

= -144 – 2 (-96) + 6(-12) = -144 + 192 – 72 = -24

21. Ans. (d): Given, $A^2 = 3A$

$\Rightarrow |A^2|=|3A| \Rightarrow |A^2|=3^3|A| \Rightarrow |A^2|-27|A|=0$

$\Rightarrow |A|[|A|-27]=0$

As, A is non – singular matrix

$\therefore |A|\neq 0 \Rightarrow |A|-27=0 \Rightarrow |A|=27$

22. Ans. (a): Given, $\begin{vmatrix}x & 0 & 8\\ 4 & 1 & 3\\ 2 & 0 & x\end{vmatrix}=0$

Expanding along R_1, we get

$x(x-0)-0+8(0-2)=0$

$\Rightarrow x^2-16=0 \Rightarrow x^2=16 \Rightarrow x=\pm 4$

23. Ans. (d): Given, A is a 3 x 3 matrix and |A| = 5

Now, $|2A'| = 2^3|A'| = 2^3|A| = 8 \times 5 = 40$

24. Ans. (b): We have, $A^T=-A$ [$\because$ A is skew-symmetric matrix]

$\therefore |A^T|=|-A| \Rightarrow |A|=(-1)^3|A|$ [$\because$ A is of order 3]

$\Rightarrow |A|=-|A| \Rightarrow 2|A|=0 \Rightarrow |A|=0$

25. Ans. (d): We have, $|3A| = 3^3|A| = 3^3.8$

[Given |A| = 8]

= 27.8 = 216

26. Ans. (b): Given, $A=\begin{bmatrix}-4 & 3\\ 7 & -5\end{bmatrix}$

$\therefore\ |A| = 20-21=-1$

And adj A = $\begin{bmatrix}-5 & -7\\ -3 & -4\end{bmatrix}^T=\begin{bmatrix}-5 & -3\\ -7 & -4\end{bmatrix}$

$\therefore\ A^{-1}=\dfrac{1}{|A|}(\text{adj}\,A)=\begin{bmatrix}5 & 3\\ 7 & 4\end{bmatrix}$

27. Ans. (a): Given, $A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 59 & 69 & -1\end{bmatrix}$

Here, |A| = -1

And adjA = $\begin{bmatrix}-1 & 0 & -59\\ 0 & -1 & -69\\ 0 & 0 & 1\end{bmatrix}^T=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ -59 & -69 & 1\end{bmatrix}$

$\therefore\ A^{-1}=\dfrac{1}{|A|}(\text{adj}\,A)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 59 & 69 & -1\end{bmatrix}=A$

28. Ans. (d): $|A|=\begin{vmatrix}1 & -2 & 4\\ 2 & -1 & 3\\ 4 & 2 & 0\end{vmatrix}$

= 1 (0 – 6) + 2(0 – 12) + 4(4 + 4) = 2 ....(i)

Given, A = adjB

$\Rightarrow\ |A|=|\text{adj}B|\Rightarrow|\text{adj}B|=2$ (Using (i))

$\Rightarrow\ |B|^2=2$ $[\because |\text{adj}B|=|B|^{3-1}$, where B is 3×3 matrix$]$

$\Rightarrow\ |B|=\pm\sqrt{2}$

$\therefore\ B^{-1}=\pm\dfrac{1}{\sqrt{2}}A\left[\because B^{-1}=\dfrac{1}{|B|}(\text{adj}B)\right]$

29. Ans. (b): We have,

$\begin{bmatrix}1 & -\tan\theta\\ \tan\theta & 1\end{bmatrix}\begin{bmatrix}1 & \tan\theta\\ -\tan\theta & 1\end{bmatrix}^{-1}=\begin{bmatrix}a & -b\\ b & a\end{bmatrix}$

Now, $\begin{bmatrix}1 & \tan\theta\\ -\tan\theta & 1\end{bmatrix}^{-1}=\dfrac{1}{\sec^2\theta}\begin{bmatrix}1 & -\tan\theta\\ \tan\theta & 1\end{bmatrix}$

$=\begin{bmatrix}\cos^2\theta & -\tan\theta\cos^2\theta\\ \cos^2\theta\tan\theta & \cos^2\theta\end{bmatrix}$

$\Rightarrow\begin{bmatrix}1 & -\tan\theta\\ \tan\theta & 1\end{bmatrix}\begin{bmatrix}\cos^2\theta & -\tan\theta\cos^2\theta\\ \cos^2\theta\tan\theta & \cos^2\theta\end{bmatrix}=\begin{bmatrix}a & -b\\ b & a\end{bmatrix}$

$\Rightarrow\begin{bmatrix}\cos^2\theta-\cos^2\theta\tan^2\theta & -2\tan\theta\cos^2\theta\\ 2\tan\theta\cos^2\theta & \cos^2\theta-\cos^2\theta\tan^2\theta\end{bmatrix}$

$=\begin{bmatrix}a & -b\\ b & a\end{bmatrix}$

$\therefore\ a=\cos^2\theta-\cos^2\theta\tan^2\theta$ and

$b=2\tan\theta\cos^2\theta$

$\Rightarrow a=\cos^2\theta\left(1-\dfrac{\sin^2\theta}{\cos^2\theta}\right)$ *and* $b=\dfrac{2\sin\theta}{\cos\theta}.\cos^2\theta$

$\Rightarrow\ a=\cos^2\theta-\sin^2\theta=\cos2\theta$ *and* $b=2\sin\theta\cos\theta=\sin2\theta$

30. Ans. (a) : We have, $|A|=\begin{vmatrix}-2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -2\end{vmatrix}$

$|A|=-2(4-0)-0+0=-8$

$\therefore |\text{adj}\,A|=(-8)^2=64$

31. Ans. (c): Given, A(adj A) = 10 I

$\Rightarrow\ |A(\text{adj})A)|=|10I|$

$\Rightarrow\ |A||\text{adj}\,A|=10^3\,|I|$ ....(i)

$\Rightarrow\ |A||A|^2=10^3$

[∵ A is matrix of order n, then |adj A| = $|A|^{n-1}$ and n = 3]

$\Rightarrow\ |A|^3=10^3$

$\Rightarrow\ |A|=10$

Now, from (i), we get

$|(\text{adj})A)|=|A|^2=10^2=100$

32. Ans. (d): We have, $\begin{vmatrix}2 & 4\\ 5 & 1\end{vmatrix}=\begin{vmatrix}2x & 4\\ 6 & x\end{vmatrix}$

$\Rightarrow\ 2-20=2x^2-24\Rightarrow 2x^2=6$

$\Rightarrow\ x^3=3\Rightarrow x=\pm\sqrt{3}$

33. Ans. (a): |AA′| = |A| |A′| = $|A'|^2=(-3)^2=9$

34. Ans. (c): ∵ A is singular matrix.

$\therefore\ \ |A|=0$

$\Rightarrow\ \begin{vmatrix}k & 8\\ 4 & 2k\end{vmatrix}=0\Rightarrow 2k^2-32=0$

$\Rightarrow k^2=16\Rightarrow k=\pm4$

35. Ans. (c): We have, $A^2 = 2A$

$\Rightarrow |A^2| = |2A|$

$\Rightarrow |A|^2 = 2^3|A|$ [As $|kA| = k^n|A|$ for a matrix of order n]

$\Rightarrow |A|[|A| - 8] = 0$

$\Rightarrow$ *either* $|A| = 0$ or $|A| = 8$

But A is non – singular matrix $\Rightarrow |A| \neq 0$

$\therefore |2A| = 2^3.|A| = 64$

36. Ans. (b): We have, |A| = -7

$$\therefore \sum_{i=1}^{3} a_{i2}A_{i2} = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} = |A| = -7$$

37. Ans. (b): Given, |A| = 5, order of matrix, n = 3.

$|adj A| = |A|^{n-1} \Rightarrow |adj A| = 25$

38. Ans. (d): We know that, $|adj A| = |A|^{n-1}$, where n is the order of A.

Here, $|adj A| = |A|^2 = (-4)^2 = 16$

39. Ans. (c): We know that, (adj A)' = cofactor matrix of A

Here, cofactor matrix of $A = \begin{bmatrix} 7 & 11 \\ -5 & 2 \end{bmatrix}$ = (adj A)'

40. Ans. (b): We have, |A| = 6 + 1 = 7

Also, $adj\ A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$\therefore 14A^{-1} = \begin{bmatrix} 4 & -2 \\ 2 & 6 \end{bmatrix}$

➤ Assertion-Reasoning (1 mark)

41. Sol. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation: Both A and R are true and R is the correct explanation of A.

42. Sol. **(d)** A is false but R is true.

Explanation: Assertion:

$$f(x) = \begin{cases} \sin \pi x, & x < 1 \\ 0, & x = 1 \\ -\frac{\sin(x-1)}{x}, & x > 1 \end{cases}$$

Also, LHL = $\lim_{x \to 1^-} f(x)$

$= \lim_{h \to 0} f(1 - h) = \lim_{h \to 0} \sin(\pi - \pi h)$

$= \lim_{h \to 0} \sin(\pi h) = \sin 0 = 0$

RHL = $\lim_{x \to 1^+} f(x)$

$= \lim_{h \to 0} f(1 + h)$

$= \lim_{h \to 0} \frac{-\sin(1+h-1)}{(1+h)}$

$= -\lim_{h \to 0} \frac{\sin h}{1+h} = 0$

and f(1) = 0

$\therefore$ LHL = RHL = f(1)

$\Rightarrow$ f(x) is continuous at x = 1.

$\therefore$ Assertion is false.

Reason: It is clear that f(1) = 0

$\therefore$ Reason is true.

43. Sol. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation: Assertion Here,

$$f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$$

For x < 0, f(x) = 2x; 0 < x < 1, f(x) = 0 and x > 1, f(x) = 4x are polynomial and constant functions, so it is continuous in the given interval.

So, we have to check the continuity at x = 0 and 1.

At x = 0

LHL = $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x)$

Putting x = 0 - h as $x \to 0^-$, $h \to 0$

$\therefore \lim_{h \to 0} [2(0 - h) = \lim_{h \to 0} (-2h) = -2 \times 0 = 0,$

RHL = $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (0) = 0$

Also, f(0) = 0

$\therefore$ LHL = RHL = f(0)

Thus, f(x) is continuous at x = 0.

At x = 1,

LHL = $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (0) = 0,$

RHL = $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x)$

Putting x = 1 + h as $x \to 1^+$ when $h \to 0$

RHL = $\lim_{h \to 0} 4(1 + h) = \lim_{h \to 0} (4 + 4h)$

= 4 + 4 × 0 = 4

$\therefore$ LHL $\neq$ RHL.

Thus, f(x) is continuous everywhere except at x = 1.

44. Sol. **(b)** Both A and R are true but R is not the correct explanation of A.

Explanation: Assertion We have, f(x) = |cos x|

$$= \begin{cases} \cos x, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Continuity at x = 0,

LHL = $\lim_{h\to 0}$ f(0 - h) = $\lim_{h\to 0}$ cos (0 - h) = cos 0 = 1

RHL = $\lim_{h\to 0}$ f(0 + h) = $\lim_{h\to 0}$ cos (0 + h)

= $\lim_{h\to 0}$ cos h = cos 0 = 1

and f(0) = 1

∴ LHL = RHL = f(0)

So, f(x) is continuous at x = 0.

Hence, f(x) is continuous everywhere.

Reason We have, f(x) = cos |x|

$$= \begin{cases} \cos x, & x \geq 0 \\ \cos(-x), & x < 0 \end{cases}$$

$$= \begin{cases} \cos x, & x \geq 0 \\ \cos x, & x < 0 \end{cases}$$

= cos x, x ∈ R

But cos x is always continuous in their domain. Hence, f(x) is continuous everywhere. Hence, both Assertion and Reason are true, but Reason is not the correct explanation of Assertion.

45. **Sol. (b)** Both A and R are true but R is not the correct explanation of A.

Explanation: Assertion: Let y = $\left(e^{\sqrt{x}}\right)^{\frac{1}{2}}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}\left(e^{\sqrt{x}}\right)^{\frac{1}{2}-1}\frac{d}{dx}e^{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(e^{\sqrt{x}}\right)^{\frac{1}{2}} \cdot e^{\sqrt{x}} \cdot \frac{d}{dx}(\sqrt{x})$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\frac{e^{\sqrt{x}}}{\sqrt{e^{\sqrt{x}}}} \times \frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{4\sqrt{x}\sqrt{e^{\sqrt{x}}}} = \frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}$$

Reason: Let y = log (log x)

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\log(\log x)) = \frac{1}{\log x}\left\{\frac{d}{dx}(\log x)\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\log x}\cdot\frac{1}{x} = \frac{1}{x\log x^2},\quad x > 1$$

Hence, both Assertion and Reason are true, but Reason is not the correct explanation of Assertion.

➢ Case Study Question

46. **Sol. (i) (d) ₹1**
Explanation: ₹1
(ii) (b) ₹5
Explanation: ₹5
(iii) (a) ₹2
Explanation: ₹2
(iv) (c) Vinod
Explanation: Vinod
(v) (a) Sunil
Explanation: Sunil

47. **Sol. (i) (b) ₹300 Explanation: ₹300**
(ii) (b) ₹500 Explanation: ₹500
(iii) (c) ₹400 Explanation: ₹400
(iv) (a) 0 **Explanation:** 0
(v) (c) |Q| **Explanation:** |Q|

➢ Questions

48. Ans. Given, $A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} \Rightarrow |A| = -12 + 12 = 0$

∴ A is a singular matrix

$$adj\ A = \begin{bmatrix} -6 & 4 \\ -3 & 2 \end{bmatrix}' = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$$

Now, A(adj A) = $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

............(i)

And $(\text{adj}\, A)A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$ ...(ii)

Similarly, $|A|I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ....(iii) $(\because |A| = 0)$

From equation (i), (ii) and (iii);

We have A(adj A) = (adj A) A = |A|I

49. Ans. We have, $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$$\therefore |A| = \begin{vmatrix} 2 & -3 \\ -4 & 7 \end{vmatrix} = 14 - 12 = 2 \neq 0$$

So, A is a non – singular matrix and therefore it is invertible.

$$\therefore\ adj\ A = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}' = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

Hence, $A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

$\Rightarrow 2A^{-1} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$ ....(i)

Now, $9I - A = 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$ (From (ii))

Hence, $2A^{-1} = 9I - A$.

50. Soln. Given, A is a skew – symmetric matrix of order 3.

So, $A = -A^T$

Now, $|A| = |-A^T|$

$|A| = (-1)^3 |A^T|$

$\left[\begin{matrix} \because |kA| = k^n |A| \\ \text{where n is of order A} \end{matrix}\right]$

$|A| = -|A| \qquad [\because |AT| = |A|]$

$|A| = -|A|$

$|A| + |A| = 0$

$\therefore \quad |A| = 0$

i.e., $\det A = 0$

51. Soln. Given,

$x - 2y = 4$

$-3x + 5y = -7$

The above system of equations can be represented in the matrix form as

$$\begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$$

Or AX = B,

Where $A = \begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix}, X \begin{bmatrix} x \\ y \end{bmatrix}$

And $B = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & -2 \\ -3 & 5 \end{vmatrix} = 5 - 6 = -1 \neq 0$

$\therefore \ A^{-1}$ exists.

Hence, the system has unique solution given by

$X = A^{-1} B$

Now $adj \ A = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$

So $A^{-1} = \frac{1}{|A|} adj \ A = \frac{1}{-1}\begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$

$= \begin{bmatrix} -5 & -2 \\ -3 & -1 \end{bmatrix}$

Now, $X = A^{-1}B$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 & -2 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ -7 \end{bmatrix}$

$= \begin{bmatrix} -20+14 \\ -12+7 \end{bmatrix}$

$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ -5 \end{bmatrix}$

$\therefore \quad x = -6 \ and \ y = -5.$

52. Soln.

Given, $A = \begin{bmatrix} 5 & 2 \\ 3 & -1 \end{bmatrix} and \ B = \begin{bmatrix} -13 \\ 1 \end{bmatrix}$

AX = B

Multiply by A^{-1}on both sides

$A^{-1}AX = A^{-1}B$

$IX = A^{-1}B \qquad [\because A^{-1}A = I]$

$\Rightarrow \quad X = A^{-1}B \qquad$ ....(i)

Now, $A = \begin{bmatrix} 5 & 2 \\ 3 & -1 \end{bmatrix}$

$|A| = \begin{vmatrix} 5 & 2 \\ 3 & -1 \end{vmatrix} = -5 - 6 = -11 \neq 0$

So A is a non – singular matrix.

As a result A^{-1} exists.

Now cofactors of A are

$C_{11} = -1, C_{12} = -3$

$C_{21} = -2, C_{22} = 5$

Then, $adj(A) = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{21} \end{bmatrix}^T$

$= \begin{bmatrix} -1 & -3 \\ -2 & 5 \end{bmatrix}^T = \begin{bmatrix} -1 & -2 \\ -3 & 5 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} adj \ A$

$= \frac{1}{-11}\begin{bmatrix} -1 & -2 \\ -3 & 5 \end{bmatrix}$

Now by equation (i)

$X = \frac{1}{-11}\begin{bmatrix} -1 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} -13 \\ 1 \end{bmatrix}$

$= \frac{1}{-11}\begin{bmatrix} 13 & -2 \\ -39 & +5 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{-1}{11}\begin{bmatrix} 11 \\ -34 \end{bmatrix} = \begin{bmatrix} -1 \\ 34/11 \end{bmatrix}$

$\therefore \quad x = -1 \ and \ y = 34/11$

[comparing corresponding elements]

$\Rightarrow \quad X = \begin{bmatrix} -1 \\ 34/11 \end{bmatrix}.$

53. Soln.

Given, $A = \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$

And $AB = I$

$$\Rightarrow A^{-1}AB = A^{-1}I$$
$$IB = A^{-1}I$$
$$B = A^{-1}I$$

Now, $A = \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & -4 \\ -1 & 2 \end{vmatrix} = 3\times 2 - (-1)\times(-4)$$
$$= 6-4$$
$$= 2 \neq 0$$

So A^{-1} can be calculated.

Cofactor of matrix A,

$C_{11} = 2,\ C_{12} = -(-1) = 1$

$C_{21} = -(-4) = 4,\ C_{22} = 3$

$$adj\ A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj}\,A)$$
$$= \frac{1}{2}\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 \\ 1/2 & 3/2 \end{bmatrix}$$

54. Soln.

Let C_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$. Then,

C_{11} = cofactor of $a_{11} = (-1)^{1+1}7 = 7$

C_{12} = cofactor of $a_{12} = (-1)^{1+2}6 = -6$

C_{21} = cofactor of $a_{13} = (-1)^{2+1}5 = -5$

C_{22} = cofactor of $a_{22} = (-1)^{2+2}1 = 1$

So adj A is

$$adj\ A = \begin{bmatrix} 6 & -6 \\ -5 & 1 \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} 7 & -5 \\ -6 & 1 \end{bmatrix}$$

55. Soln.

Given, $A'A = I$

Then $|A'A| = |I|$

$\Rightarrow |A'||A| = |I|$

$\Rightarrow |A||A| = |I| \quad [\because |A'| = |A|]$

$\Rightarrow |A|^2 = 1 \quad [\because |I| = 1]$

$\Rightarrow |A| = \pm 1 \quad$ Ans.

56. Soln.

Since $|kA| = k^n|A|$, where n is the order of matrix. $= 27\,|A|$

$\therefore \quad |3A| = 3^3(A) \quad$ or $\quad k = 27$

57. Soln.

$$A = \begin{pmatrix} 4 & 6 \\ 7 & 5 \end{pmatrix}$$

$$adj\ A = \begin{pmatrix} 5 & -6 \\ -7 & 4 \end{pmatrix}$$

$$A.(\text{adj}\,A) = \begin{pmatrix} 4 & 6 \\ 7 & 5 \end{pmatrix}\begin{pmatrix} 5 & -6 \\ -7 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} 20-42 & -24+24 \\ 35-35 & -42+20 \end{pmatrix}$$
$$= \begin{pmatrix} -22 & 0 \\ 0 & -22 \end{pmatrix}$$

58. Soln.

$$A = \begin{vmatrix} 3-2x & x+1 \\ 2 & 4 \end{vmatrix}$$

Since A is a singular matrix. i.e., |A| = 0

$$\begin{vmatrix} 3-2x & x+1 \\ 2 & 4 \end{vmatrix} = 0$$

Or, $4(3-2x)-2(x+1) = 0$

Or, $12-8x-2x-2 = 0$

Or, $-10x+10 = 0$

Or, x = 1

59. Soln. Since, A is a singular matrix

$$\begin{vmatrix} 1+x & 7 \\ 3-x & 8 \end{vmatrix} = 0$$

Or, $8(1+x)-7(3-x) = 0$

Or, $8+8x-21+7x = 0$

Or, $15x-13 = 0$

Or $\quad x = \frac{13}{15}$

60. Soln.

We know that

$(AB)\text{adj}(AB) = |AB| = \text{adj}(AB)(AB)$ ....(i)

$(AB)(\text{adj}\,B\,\text{adj}\,A)$

$= A.B\,adj\,B.adj\,A = A(B\,\text{adj}\,B)\,\text{adj}\,A$

$$= A(|B|I)\,\mathrm{adj}\,A \quad [\because B\,\mathrm{adj}\,B = |B|I]$$
$$= |B|(A.\mathrm{adj}\ A)$$
$$= |B||A|I \quad [\because A\,\mathrm{adj}\,A = |A|I]$$
$$= |A||B|I$$
$$= |AB|I \quad \ldots\ldots\text{(ii)}$$

From (i) and (ii), we get

$AB(\mathrm{adj}\,AB) = AB(\mathrm{adj}\,B.\mathrm{adj}\,A)$

Pre-multiplying both sides by $(AB)^{-1}$, we get

$$(AB)^{-1}[(AB)\,\mathrm{adj}\,AB]$$
$$= (AB)^{-1}[(AB)\,\mathrm{adj}\,B.\mathrm{adj}\,A]$$

Or $\quad adj\ AB = adj\ B.adjA$

61. Soln. We have,

$$A = \Delta\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Clearly, the cofactors of elements of |A| are given by

$A_{11} = \cos\alpha;\ A_{12} = -\sin\alpha;\ A_{13} = 0;$

$A_{21} = \sin\alpha;\ A_{22} = \cos\alpha;\ A_{23} = 0$

$A_{31} = 0;\ A_{32} = 0\ and\ \ A_{33} = 1$

$$\therefore adj(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, $A(\mathrm{adj}\,A)$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\alpha + \sin^2\alpha & 0 & 0 \\ 0 & \sin^2\alpha + \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \ldots\text{.(i)}$$

$$(\mathrm{adj}\,A).(A) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\alpha + \sin^2\alpha & 0 & 0 \\ 0 & \sin^2\alpha + \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \ldots\ldots\text{(ii)}$$

$$\text{And } |A| = \begin{vmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1.(\cos^2\alpha + \sin^2\alpha) = 1 \quad \ldots\ldots\ldots\text{(iii)}$$

From eqs. (i), (ii) and (iii), we get

$A(\mathrm{adj}\,A) = (\mathrm{adj}\,A).A = |A|I_3$

62. Soln.

$|A| = 3(3 - 6) + (-2)(-12 - 14) + 1(12 + 7) = 62 \neq 0$

Hence, A^{-1} exists. Let c_{ij} represent the cofactor of (i, j)th element of A. Then,

$c_{11} = 3, c_{12} = 26, c_{13} = 19, c_{21} = 9, c_{22} = -16, c_{23} = 5$
$, c_{31} = 5, c_{32} = -2, c_{33} = -11.$

$$adj\ A = \begin{bmatrix} -3 & 9 & 5 \\ 26 & -16 & -2 \\ 19 & 5 & -11 \end{bmatrix}$$

$$A^{-1} = \frac{1}{62}\begin{bmatrix} -3 & 9 & 5 \\ 26 & -16 & -2 \\ 19 & 5 & -11 \end{bmatrix}$$

Then given system of equations is equivalent to the matrix equation

$$AX = B, where\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 14 \\ 4 \\ 0 \end{bmatrix}$$

Or $\quad X = A^{-1}B$

$$= \frac{1}{62}\begin{bmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{bmatrix}\begin{bmatrix} 14 \\ 4 \\ 0 \end{bmatrix}$$

$$= \frac{1}{62}\begin{bmatrix} 62 \\ 62 \\ 62 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence, x = 1, y = 1, z = 1

63. Soln.

Given $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & -3 & 1 \end{bmatrix}$

$|A| = 1(1+3) - 2(-1-1) + 1(3-1)$

$= 4+4+2 = 10$

Co-factor Matrix is:

$$= \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$\therefore$ $adj\ A =$ transpose of above matrix

$$= \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{adj(A)}{|A|}$$

$$= \frac{1}{10}\begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 2/5 & -1/2 & 1/10 \\ 1/5 & 0 & -1/5 \\ 1/5 & 1/2 & 3/10 \end{bmatrix}$$

Given set of equations are:

$x + 2y + z = 4$

$-x + y + z = 0$

$x - 3y + z = 4$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}$$

$\Rightarrow \quad AX = B$

Multiplying both sides by A^{-1}, we get

$A^{-1}AX = A^{-1}B$

$\Rightarrow \quad X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2/5 & -1/2 & 1/10 \\ 1/5 & 0 & -1/5 \\ 1/5 & 1/2 & 3/10 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{8}{5} + \frac{2}{5} \\ \frac{4}{5} - \frac{4}{5} \\ \frac{4}{5} + \frac{6}{5} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$\therefore \quad x = 2;\ y = 0,\ z = 2$

64. Soln.

Given $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

And $B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$

$$\therefore AB = \begin{bmatrix} 2+4 & 2-2 & -4+4 \\ 4-12+8 & 4+6-4 & -8-12+20 \\ -4+4 & 2-2 & -4+10 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6I$$

$\therefore \quad AB = 6I$

Premultiplying by A^{-1}

$A^{-1}AB = 6A^{-1}I$

$\Rightarrow \quad IB = 6A^{-1}I \qquad (\because A^{-1}A = I)$

$\Rightarrow \quad B = 6A^{-1} \qquad (\because IX = X)$

$\Rightarrow \quad A^{-1} = \frac{1}{6}B$

Given equations are:

$x - y = 3$

$2x + 3y + 4z = 17$

$y + 2z = 7$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$AX = C, where\, C = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \quad A^{-1}AX = A^{-1}C$$

$$\Rightarrow \quad X = A^{-1}C$$

$$\Rightarrow \quad X = \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 12 \\ -6 \\ 24 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

$$\therefore \quad x = 2,\ y = -1\, and\, z = 4$$

65. Soln.

Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$= \begin{vmatrix} a+b+c & b & c \\ b+c+a & c & a \\ c+a+b & a & b \end{vmatrix}$ [Applying $C_1 \to C_1 + C_2 + C_3$]

$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$ [Taking out (a + b + c) from C₁]

$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$ [Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$= (a+b+c)\begin{vmatrix} c-b & a-c \\ a-b & b-c \end{vmatrix}$ [Expanding along C₁]

$= (a+b+c)\{-(b-c)^2 - (a-c)(a-b)\}$

LHS = $-(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

Also, RHS =

$-(a+b+c)(a+bw+cw^2)(a+bw^2+cw)$

$= -(a+b+c)(a^2+abw^2+acw+abw+b^2w^2+$
$bcw^2+acw^2+bcw^4+c^2w^3)$

$= -(a+b+c)[(a^2+b^2+c^2+ab(w^2+w)+$
$bc(w^2+w^4)+ca(w+w^2)]$

$= -(a+b+c)(a^2+b^2+c^2-ab-bc-ca) = \text{LHS}$

66. Soln.

Let

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin cC+\sin^2 C \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & \sin B-\sin A & \sin C-\sin A \\ \sin A+\sin^2 A & \sin^2 B-\sin^2 A+\sin B-\sin A & \sin^2 C-\sin^2 A+\sin C-\sin A \end{vmatrix}$$

$$\begin{bmatrix} Applying\ C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{bmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & \sin B-\sin A & \sin C-\sin A \\ \sin A+\sin^2 A & (\sin B-\sin A)(\sin B+\sin A+1) & (\sin C-\sin A)(\sin C+\sin A+1) \end{vmatrix}$$

$$= (\sin B-\sin A)(\sin C-\sin A)\begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & 1 & 1 \\ \sin A+\sin^2 A & \sin B+\sin A+1 & \sin C+\sin A+1 \end{vmatrix}$$

Expanding along R₁, we get

$(\sin B-\sin A)(\sin C-\sin A)[\sin C+\sin A+1-\sin B-\sin A-1]$

$= (\sin B-\sin A)(\sin C-\sin A)(\sin C-\sin B)$

$\therefore \quad \Delta = 0$

$\Rightarrow (\sin B-\sin A)(\sin C-\sin B)(\sin C-\sin A) = 0$

$\Rightarrow \sin B-\sin A = 0\ or\ \sin C-\sin B = 0\ or\ \sin C-\sin A = 0$

$\Rightarrow B = A\ or\ C = B\ or\ C = A$

$\Rightarrow \quad \Delta ABC$ is an isosceles triangle.

67. Soln. Let the cost of varieties of pens A, B and C be x, y and z respectively.

From question

$$x+y+z = 21$$
$$4x+3y+2z = 60$$
$$6x+2y+3z = 70$$

The given system of linear equation in matrix equation is as follows

AX = B, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 6 & 2 & 3 \end{bmatrix},\quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} and\ B = \begin{bmatrix} 21 \\ 60 \\ 70 \end{bmatrix}$$

$\because \quad AX = B \quad \Rightarrow X = A^{-1}B \quad$ ....(i)

Now

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 6 & 2 & 3 \end{vmatrix} = 1(9-4) - 1(12-12)$$
$$+(8-18) = 5-0-10 = -5 \neq 0$$

$A_{11} = 9-4 = 5 \quad A_{21} = -(3-2) = -1 \quad A_{31} = 2-3 = -1$

$A_{12} = -(12-12) = 0 \quad A_{22} = (3-6) = -3 \quad A_{32} = -(2-4) = 2$

$A_{13} = (8-18) = -10 \quad A_{23} = -(2-6) = 4 \quad A_{33} = (3-4) = -1$

$$\therefore\ Adj\ A = \begin{bmatrix} 5 & 0 & -10 \\ -1 & -3 & 4 \\ -1 & 2 & -1 \end{bmatrix}' = \begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{|A|} adj\ A = -\frac{1}{5}\begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}$$

Now from (i) $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}\begin{bmatrix} 21 \\ 60 \\ 70 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -25 \\ -40 \\ -40 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$\Rightarrow x = 5,\ y = 8,\ z = 8$

$\Rightarrow \quad Cost\ of\ pen\ A = Rs.5$

$\quad Cost\ of\ pen\ B = Rs.8$

$\quad Cost\ of\ pen\ C = Rs.8$

68. Soln. Let x, y and z be the awarded money for honesty, regularity and hardwork.

According to question, the system of equations are

$x + y + z = 6000 \quad .....(i)$

$x + 3z = 11000 \quad ......(ii)$

$x - 2y + z = 0 \quad ....(iii)$

The given situation may be written in matrix form as AX = B,

$$\text{Where } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$$

$$\text{Now, } |A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix} = 1(0+6) - 1(1-3) + 1$$

$$(-2-0) = 6 + 2 - 2 = 6 \neq 0$$

Hence, A^{-1} exists and system have unique solution.

If A_{ij} is co-factor of a_{ij}, then

$A_{11} = (-1)^{1+1}\begin{vmatrix} 0 & 3 \\ -2 & 1 \end{vmatrix} = 0 + 6 = 6;$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix} = (-2-0) = -2;$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0;$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = 3 - 0 = 3;$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = -(1-3) = 2;$

$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = -(1+2) = -3$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -(-2-1) = 3$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -(3-1) = -2$

$$adj\ A = \begin{bmatrix} 6 & 2 & -2 \\ -3 & 0 & 3 \\ 3 & -2 & -1 \end{bmatrix}^T = \begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adj\ A$$

$$= \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

Putting the value of X, A^{-1} and B in X = A^{-1} B, we get

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 36000 - 33000 + 0 \\ 12000 + 0 + 0 \\ -12000 + 33000 + 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3000 \\ 12000 \\ 21000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 500 \\ 2000 \\ 3500 \end{bmatrix}$$

$\Rightarrow \quad x = 500, \quad y = 2000, \quad z = 3500$

69. Soln.

Using determinants, the line joining A(1, 3) and B(0, 0) is given by $\begin{vmatrix} x & y & 1 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0$

$\Rightarrow \quad 1(3x - y) = 0 \Rightarrow y = 3x$

Now, D(k, 0) is a point s.t. area of $\Delta ABD = 3\,sq.$ units

$$\therefore \quad \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = 3$$

$\Rightarrow \quad (0 + 3k) = \pm 6 \Rightarrow k = \pm 2$

70. Soln.

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$$

$$|A'| = \begin{vmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{vmatrix} = 1(-1-8) - 2(-8+3)$$

$= -9 + 10 = 1 \neq 0$. So, $(A')^{-1}$ exists.

Let the cofactors of a_{ij}'s are A_{ij} in A'

Now, $A_{11} = -9,\ A_{12} = 8,\ A_{13} = -5,$

$A_{21} = -8,\ A_{22} = 7,\ A_{23} = -4,$

$A_{31} = -2,\ A_{32} = 2,\ A_{33} = -1$

$$\therefore \quad adj(A') = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

$$\therefore (A')^{-1} = \frac{adj(A')}{|A'|} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

71. Soln.

Here, $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$$\Rightarrow \quad |A| = \begin{vmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix}$$

$= -1(1-4) - (-2)(2+4) - 2(-4-2)$

$= 3 + 12 + 12 = 27$

Now, $A_{11} = -3,\ A_{12} = -6,\ A_{13} = -6,$

$A_{21} = 6,\ A_{22} = 3,\ A_{23} = -6$

$A_{31} = 6,\ A_{32} = -6,\ A_{33} = 3$

$$\therefore adj\ A = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$\therefore \quad A.(\text{adj}\,A) = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} . \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix} = 27\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\,I_3$$

72. Soln.

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$\Rightarrow \quad A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$ ....(i)

Now, $4A - 3I = 4\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \quad 4A - 3I = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$ ....(ii)

From (i) and (ii), we get

$A^2 = 4A - 3I$

Pre-multiplying by A^{-1} on both sides, we get

$A^{-1}(A^2) = 4A^{-1}A - 3A^{-1}I$

$\Rightarrow \quad A = 4I - 3A^{-1} \quad [\because AA^{-1} = I]$

$\Rightarrow \quad 3A^{-1} = 4I - A$

$\Rightarrow \quad A^{-1} = \frac{4}{3}I - \frac{1}{3}A$

$$= \frac{4}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

73. Soln.

Given, $A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now, $A_{11} = \cos\alpha, A_{12} = -\sin\alpha, A_{13} = 0,$

$A_{21} = \sin\alpha, A_{22} = \cos\alpha, A_{23} = 0,$

$A_{31} = 0, A_{32} = 0, A_{33} = 1$

$$\therefore\ adj(A)=\begin{bmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$A.\text{adj}(A)=\begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad \text{.............(i)}$$

$$adj(A).A=\begin{bmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad \text{..........(ii)}$$

$$|A|=\begin{vmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{vmatrix}$$

$$=\cos\alpha(\cos\alpha-0)+\sin\alpha(\sin\alpha-0)+0=1$$

.....(iii)

From (i), (ii) and (iii), we get

$$A(\text{adj A})=(\text{adj A})\,A=|A|\,I_3.$$

74. Soln.

We have, $$A=\begin{bmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{bmatrix}$$

$$\Rightarrow\ |A|=\begin{vmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{vmatrix}$$

$$=2(-4+4)+3(-6+4)+5(3-2)$$

$$=-6+5=-1\neq 0$$

$\therefore\ A^{-1}$ *exists.*

Now, $A_{11}=0,\ A_{12}=2,\ A_{13}=1,\ A_{21}=-1,\ A_{22}=-9,$
$A_{23}=-5,\ A_{31}=2,\ A_{32}=23,\ A_{33}=13$

$$\therefore\quad adj\ A=\begin{bmatrix}0 & -1 & 2\\ 2 & -9 & 23\\ 1 & -5 & 13\end{bmatrix}$$

Now, $$A^{-1}=\frac{1}{|A|}adj\ A=(-1)\begin{bmatrix}0 & -1 & 2\\ 2 & -9 & 23\\ 1 & -5 & 13\end{bmatrix}$$

$$=\begin{bmatrix}0 & 1 & -2\\ -2 & 9 & -23\\ -1 & 5 & -13\end{bmatrix}$$

The given system of equation is

$2x-3y+5z=11,\ 3x+2y-4z=-5,\ x+y-2z=-3$

The system of equations can be written as AX = B,

where $$A=\begin{bmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}, B=\begin{bmatrix}11\\ -5\\ -3\end{bmatrix}$$

Since A^{-1} exists, therefore, system of equations has a unique solution given by

$$X=A^{-1}B=\begin{bmatrix}0 & 1 & -2\\ -2 & 9 & -23\\ -1 & 5 & -13\end{bmatrix}\begin{bmatrix}11\\ -5\\ -3\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$$

$$\Rightarrow\quad x=1,\ y=2\ and\ z=3.$$

SURE SHOT QUESTIONS

ARVIND ACADEMY

Chapter – 05 (Solution)

Continuity and Differentiability

➢ MCQ (1 mark)

1. Soln. (d): $f(x)=2x$ and $g(x)=\frac{x^2}{2}+1$ both are continuous function.

So the function $\frac{g(x)}{f(x)}$ can be discontinuous.

If, $f(x)=0 \Rightarrow 2x=0 \Rightarrow x=0$

Hence, at x = 0, $\frac{g(x)}{f(x)}$ is discontinuous.

2. Soln. (c): $f(x)=\frac{4-x^2}{4x-x^3}=\frac{4-x^2}{x(2-x)(2+x)}$

So, f(x) is discontinuous at x = 0, 2, -2.

3. Soln. (b): $f(x)=|2x-1|\sin x$

$$f(x)=\begin{cases}-(2x-1)\sin x, & x<1/2\\ (2x-1)\sin x, & x\ge 1/2\end{cases}$$

$$Lf'\left(\frac{1}{2}\right)=\lim_{h\to 0}\frac{f(1/2-h)-f(1/2)}{-h}$$

$$=\lim_{h\to 0}\frac{-[2(1/2-h)-1]\sin(1/2-h)-0}{-h}$$

$$=\lim_{h\to 0}\frac{-2h\sin(1/2-h)}{h}=-2\sin(1/2)$$

$$Rf'(1/2)=\lim_{h\to 0}\frac{f(1/2+h)-f(1/2)}{h}$$

$$=\lim_{h\to 0}\frac{[2(1/2+h)-1]\sin(1/2+h)-0}{h}$$

$$=\lim_{h\to 0}\frac{2h\sin(1/2+h)}{h}=2\sin(1/2)$$

$Lf'(1/2)\ne Rf'(1/2)$

So, f(x) is not differentiable at x = ½

Hence, f(x) is differentiable $\forall\ x\in R-\left\{\frac{1}{2}\right\}$

4. Soln. (a): f(x) = cotx is discontinuous if $\cot x\to\infty$

$\Rightarrow \cot x=\cot 0 \Rightarrow x=n\pi$ and $\forall\ n\in Z.$

5. Soln. (a): $f(x)=e^{|x|}$; $f(x)=\begin{cases}e^{-x}, & x<0\\ e^{x}, & x\ge 0\end{cases}$

$\therefore$ e^x is continuous for all x > 0 and e^{-x} is continuous for all x < 0.

Now, $\lim_{x\to 0^-}f(x)=\lim_{x\to 0}e^{-x}=e^0=1$

$\lim_{x\to 0^+}f(x)=\lim_{x\to 0}e^{x}=e^0=1$

$\lim_{x\to 0^-}f(x)=\lim_{x\to 0^+}f(x)=\lim_{x\to 0}f(x)=f(0)$

Hence, f(x) is continuous everywhere.

$$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$=\lim_{h\to 0}\frac{e^{-(-h)}-1}{-h}=\lim_{h\to 0}\frac{\left(1+h+\frac{h^2}{2!}+\frac{h^3}{3!}+\ldots\right)-1}{-h}$$

$$=\lim_{h\to 0}\left(-1-\frac{h}{2!}-\frac{h^2}{3!}+\ldots\right)=-1$$

$$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{e^h-1}{h}$$

$$=\lim_{h\to 0}\frac{\left(1+h+\frac{h^2}{2!}+\frac{h^3}{3!}+\ldots\right)-1}{h}=\lim_{h\to 0}\left(1+\frac{h}{2!}+\frac{h^2}{3!}+\ldots\right)=1$$

$L(f'(0))\ne R(f'(0))$

SO f(x) is not differentiable at x = 0.

6. Soln. (a): $f(x) = x^2 \sin\frac{1}{x}$ $for\ x \neq 0$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin\frac{1}{x}$$

Since the value of $\sin\frac{1}{x}$ is between -1 to 1.

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = (0)^2 \times (\text{value between} -1 \text{ to } 1) = 0$$

So $f(x)$ is continuous at x = 0

If $\lim_{x\to 0} f(x) = f(0) \Rightarrow f(0) = 0$

7. Soln. (c): $f(x) = \begin{cases} mx+1, & if\ x \leq \frac{\pi}{2} \\ \sin x + n, & if\ x > \frac{\pi}{2} \end{cases}$

$$\lim_{x\to \frac{\pi}{2}^-} f(x) = \lim_{x\to \frac{\pi}{2}} (mx+1) = \frac{m\pi}{2} + 1$$

$$\lim_{x\to \frac{\pi}{2}^+} f(x) = \lim_{x\to \frac{\pi}{2}} (\sin x + n) = \sin\frac{\pi}{2} + n = 1 + n$$

Since the function is continuous at $x = \frac{\pi}{2}$

$$\Rightarrow \frac{m\pi}{2} + 1 = 1 + n \Rightarrow n = \frac{m\pi}{2}$$

8. Soln. (b): From the graph of f(x) = |sinx|, it is clear that f(x) is continuous everywhere but not differentiable at $x = n\pi,\ n \in Z$.

9. Soln. (b):

$$y = \log\left(\frac{1-x^2}{1+x^2}\right) \Rightarrow y = \log(1-x^2) - \log(1+x^2)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{1-x^2}(-2x) - \frac{1}{1+x^2}(2x) = \frac{-4x}{1-x^4}$$

10. Soln. (a): $y = \sqrt{\sin x + y} \Rightarrow y^2 = \sin x + y$

Differentiating w.r.t. to x, we get

$$2y\frac{dy}{dx} = \cos x + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y-1}$$

11. Soln. (a): Let $y = \cos^{-1}(2x^2-1) = 2\cos^{-1} x$

Differentiating w.r.t. $\cos^{-1} x$, we get

$$\frac{dy}{d(\cos^{-1}x)} = \frac{2d(\cos^{-1}x)}{d(\cos^{-1}x)} = 2$$

12. Soln. (b): $x = t^2$ .....(i), $y = t^3$ ....(ii)

From (i) and (ii), we get $y = x^{3/2}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$$

Again differentiating w.r.t x, we get

$$\frac{d^2y}{dx^2} = \frac{3}{4\sqrt{x}} = \frac{3}{4t} \qquad \text{(From (i))}$$

13. Soln. ([illegible]): Since Rolle's theorem satisfy in $[0, \sqrt{3}]$.

$$\therefore f'(c) = 0 \Rightarrow 3c^2 - 3 = 0$$

$$\Rightarrow c^2 = 1 \Rightarrow c = \pm 1 \Rightarrow c = 1$$

$$(\because c = -1 \notin [0, \sqrt{3}])$$

14. Soln. (b):

$$f'(c) = \frac{f(3)-f(1)}{3-1} \Rightarrow 1 - \frac{1}{c^2} = \left(\frac{\frac{10}{3} - 2}{2}\right)$$

$$\Rightarrow 1 - \frac{1}{c^2} = \frac{4}{3\times 2} \Rightarrow \frac{1}{c^2} = 1 - \frac{2}{3} \Rightarrow c^2 = 3 \Rightarrow c = \pm\sqrt{3}$$

$$\Rightarrow c = \sqrt{3} \qquad (\because -\sqrt{3} \notin [1,3])$$

15. Ans. (b): Let x = 1.5

$\therefore$ L.H.L. = $\underset{x\to 1.5^-}{Lt} f(x) = \underset{h\to 0}{Lt}[1.5 - h] = 1$

And R.H.L. = $\underset{x\to 1.5^+}{Lt} f(x) = \underset{h\to 0}{Lt}[1.5 + h] = 1$

$\because$ $L.H.L. = R.H.L.$

$\therefore$ f(x) is continuous at x = 1.5

Also, greatest integer function is discontinuous at all integral values of x.

16. Ans. (b): Since f(x) is continuous at x = 5,

$$\Rightarrow \lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = f(5)$$

$$\Rightarrow 3(5)-8 = 2k \Rightarrow 7 = 2k \Rightarrow k = \frac{7}{2}$$

17. Ans. (c): Since, greatest integer function i.e., [x] is continuous at all points except at integers.

$\therefore$ f(x) is continuous at 1.5.

18. Ans. (c): $f(x) = |x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$

The function f(x) is continuous everywhere but not differentiable at x = 0 as at x = 0

$$Lf'(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{-x-0}{x} = -1$$

$$Rf'(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x-0}{x} = 1$$

$\therefore \quad Lf'(0) \neq Rf'(0)$, so f(x) is not differentiable at x = 0.

19. Ans. (c): Let y = x^{2x}

Taking log on both sides, we get

$$\log y = 2x \log x$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = 2\left\{x.\frac{1}{x} + \log x.1\right\}$$

$$\Rightarrow \frac{dy}{dx} = 2y\{1+\log x\} = 2x^{2x}(1+\log x)$$

20. Ans. (a): Given, $y^2(2-x) = x^3$

$$\Rightarrow y^2 = \frac{x^3}{2-x} \Rightarrow 2y.\frac{dy}{dx} = \frac{(2-x)\times 3x^2 - x^3(-1)}{(2-x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{6x^2-2x^3}{2y(2-x)^2} \Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = \frac{6-2}{2\times 1} = 2$$

21. Ans. (a): At x = 1 $\lim_{x\to 1^-} f(x) = \lim_{x\to 1} x^2 = 1$

And $\lim_{x\to 1^+} f(x) = \lim_{x\to 1} 2-x = 1$

Also,

$$f(1) = 2-1 = 1 \because \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(x)$$

$\therefore$ f(x) is continuous at x = 1

Now, L.H.D. =

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 2$$

R.H.D. = $\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{(2-x)-1}{x-1} = -1$

$\because$ $L.H.D. \neq R.H.D.$ $\therefore$ f(x) is not differentiable at x = 1.

22. Ans. (c): Given, $\sec^{-1}\left(\frac{1+x}{1-y}\right) = a \Rightarrow \sec a = \frac{1+x}{1-y}$

On differentiating, we get

$$\frac{(1-y)+(1+x)\frac{dy}{dx}}{(1-y)^2} = 0 \Rightarrow (1+x)\frac{dy}{dx} = y-1$$

$$\Rightarrow \frac{dy}{dx} = \frac{y-1}{1+x}$$

23. Ans. (c): $y = \log(\sin e^x)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sin e^x}.\frac{d}{dx}(\sin e^x) = \frac{1}{\sin e^x}\cos e^x.\frac{d}{dx}e^x = \frac{1}{\sin e^x}\cos ec^x.e^x$$

$$= e^x \cot e^x$$

24. Ans. (a): Given, $y = \tan^{-1}\left(e^{2x}\right)$

$$\therefore \frac{dy}{dx} = \frac{1}{1+e^{4x}} \times 2e^{2x} = \frac{2e^{2x}}{1+e^{4x}}$$

25. Ans. (d): We have, $x = A\cos 4t + B\sin 4t$

Differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt} = A.(-\sin 4t).4 + B\cos t\,4t.4 \text{ ....(i)}$$

Again differentiating both sides of (i) w.r.t. t, we get

$$\frac{d^2x}{dt^2} = -4A(\cos 4t).4 + 4B(-\sin 4t).4$$

$$= -16A\cos 4t - 16B\sin 4t = -16(A\cos 4t + B\sin 4t)$$

$$= -16x$$

26. Ans. (b): Given, $y = e^{-x}$

$$\Rightarrow \frac{dy}{dx} = -e^{-x} \Rightarrow \frac{d^2y}{dx^2} = e^{-x} = y$$

27. Ans. (b): Given, x = t² + 1 and y = 2at

$$\Rightarrow \frac{dx}{dt} = 2t \Rightarrow \frac{dy}{dt} = 2a \therefore \frac{dy}{dx} = \frac{a}{t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-a}{t^2}.\frac{dt}{dx} = \frac{-a}{2t^3} \therefore \left(\frac{d^2y}{dx^2}\right)_{at\, t=a}$$

$$= \frac{-a}{2a^3} = \frac{-1}{2a^2}$$

28. Ans. (c): We have, $y = \sin(2\sin^{-1}x)$

$$\Rightarrow y = \sin\left[\sin^{-1}\left(2x\sqrt{1-x^2}\right)\right]$$

$$\left[\because 2\sin^{-1}x = \sin^{-1}2x\sqrt{1-x^2}\right]$$

$$\Rightarrow y = 2x \times \sqrt{1-x^2} \quad ....(i)$$

$$\Rightarrow y_1 = 2x \times \frac{-2x}{2\sqrt{1-x^2}} + 2\sqrt{1-x^2} = \frac{-4x^2+2}{\sqrt{1-x^2}}$$

......(ii)

$$\therefore y_2 = \frac{\sqrt{1-x^2}(-8x) - (-4x^2+2) \times \frac{-2x}{2\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{4x^3-6x}{(1-x^2)\sqrt{1-x^2}} \Rightarrow (1-x^2)y_2 = \frac{4x^3-6x}{\sqrt{1-x^2}}$$

Now, consider $xy_1 - 4y$

$$= \frac{-4x^3+2x}{\sqrt{1-x^2}} - 8 \times \sqrt{1-x^2} \quad \text{[Using (i) and (ii)]}$$

$$= \frac{4x^3-6x}{\sqrt{1-x^2}}$$

Thus, $(1-x^2)y_2 = xy_1 - 4y$

29. Ans. (d): We have, $y = \log_e\left(\frac{x^2}{e^2}\right)$

$$\therefore \frac{dy}{dx} = \frac{e^2}{x^2}.\frac{1}{e^2}.2x = \frac{2}{x} \Rightarrow \frac{d^2y}{dx^2} = -\frac{2}{x^2}$$

30. Ans. (c): Given, the function f is continuous at x = 0.

$$\therefore \lim_{x\to 0} f(x) = f(0).$$

Now, $\lim_{x\to 0} f(x)$

$$= \lim_{x\to 0}\frac{1-\cos 4x}{8x^2} = \lim_{x\to 0}\frac{2\sin^2 2x}{8x^2} = \lim_{x\to 0}\frac{\sin^2 2x}{4x^2}$$

$$= \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2 = 1$$

Also, f(0) = k

Hence, k = 1

31. Ans. (b): We have, $f(x) = \begin{cases} \frac{1-\cos kx}{x\sin x}, & x \neq 0 \\ \frac{1}{2}, & x = 0 \end{cases}$

∵ f(x) is continuous at x = 0.

$$\therefore \lim_{x\to 0}\frac{1-\cos kx}{x\sin x} = \frac{1}{2} \Rightarrow \lim_{x\to 0}\frac{2\sin^2\frac{kx}{2}}{x^2\frac{\sin x}{x}} = \frac{1}{2}$$

$$\Rightarrow \lim_{x\to 0} 2.\frac{k^2}{4}\left\{\frac{\sin\left(\frac{kx}{2}\right)}{\frac{kx}{2}}\right\}^2 \frac{1}{\frac{(\sin x)}{x}} = \frac{1}{2}$$

$$\Rightarrow \frac{k^2}{2} = \frac{1}{2} \Rightarrow k = \pm 1$$

But k < 0 ∴ k = -1

32. Ans. (a): We have, $f(x) = \begin{cases} \frac{x}{|x|}, & x < 0 \\ -1, & x \geq 0 \end{cases}$

$$\Rightarrow f(x) = \begin{cases} \frac{x}{-x} = -1, & x < 0 \\ -1, & x \geq 0 \end{cases}$$

$$\Rightarrow f(x) = -1 \,\forall\, x \in R$$

⇒ f(x) is continuous $\forall\ x \in R$ as it is a constant function.

33. Ans. (c): We have, $e^x + e^y = e^{x+y}$

$$\Rightarrow e^{-y} + e^{-x} = 1$$

Differentiating w.r.t. x, we get

$$-e^{-y}\frac{dy}{dx} - e^{-x} = 0 \Rightarrow \frac{dy}{dx} = -e^{y-x}$$

34. Ans. (d): We have, $y = \log(\cos e^x)$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\cos e^x}.(-\sin e^x).e^x$$

$$\Rightarrow \frac{dy}{dx} = -e^x \tan e^x$$

35. Ans. (a): Let $u = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ ....(i)

And $v = \sin^{-1} x, \frac{1}{\sqrt{2}} < x < 1$

$\Rightarrow \sin v = x$ ....(ii)

From (i) and (ii), we get

$u = \sin^{-1}(2\sin v \cos v) = \sin^{-1}(\sin 2v)$

$\Rightarrow u = 2v$

Differentiating with respect to v both sides, we get

$$\frac{du}{dv} = 2$$

36. Ans. (a): We have, y = 5 cosx – 3 sunx

$$\Rightarrow \frac{dy}{dx} = -5\sin x - 3\cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = -5\cos x + 3\sin x = -y$$

37. Ans. (a): We have, $x = a\sec\theta$

$$\Rightarrow \frac{dx}{d\theta} = a\tan\theta\sec\theta \; and \; y = b\tan\theta \Rightarrow \frac{dy}{d\theta} = b\sec^2\theta$$

$$\therefore \frac{dy}{dx} = \frac{b\sec^2\theta}{a\tan\theta\sec\theta} = \frac{b}{a}\cos ec\theta$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-b}{a}\cos ec\theta\cot\theta\frac{d\theta}{dx}$$

$$= \frac{-b}{a}\cos ec\theta\cot\theta\frac{1}{a\tan\theta\sec\theta} = \frac{-b}{a^2}\cot^3\theta$$

$$\therefore \left[\frac{d^2y}{dx^2}\right]_{\theta=\frac{\pi}{6}} = \frac{-3\sqrt{3}b}{a^2}$$

➢ Assertion-Reasoning (1 mark)

38. Sol. **(c)** A is true but R is false.

Explanation: Assertion:

$$f(x) = |x - 3| = \begin{cases} x - 3, & x \geq 3 \\ 3 - x, & x < 3 \end{cases}$$

$$\therefore \text{LHL} = \lim_{x\to0^-} f(x) = \lim_{x\to0^-} f(0 - h)$$

$$= \lim_{h\to0}(3 + h) = 3$$

$$\text{RHL} = \lim_{x\to0^+} f(x) = \lim_{h\to0} f(0 + h)$$

$$= \lim_{h\to0}(3 - h) = 3$$

and f(0) = 3 - 0 = 3

⇒ LHL = RHL = f(0)

So, f(x) is continuous at x = 0.

Reason: At x = 3 L.H.D $= \lim_{x\to3}\frac{f(x)-f(3)}{x-3}$

$$= \lim_{x\to3}\frac{(3-x)-0}{x-3}$$

$$= \lim_{x\to3}(-1)$$

= (-1)

At the same point RHD $= \frac{f(x)-f(3)}{x-3}$

$$= \lim_{x\to3}\frac{(x-3)-0}{x-3}$$

= 1

So at x = 3, LHD≠RHD

39. Sol. **(b)** Both A and R are true but R is not the correct explanation of A.

Explanation: Assertion We have, f(x) = |cos x|

$$= \begin{cases} \cos x, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Continuity at x = 0,

$$\text{LHL} = \lim_{h\to0} f(0 - h) = \lim_{h\to0}\cos(0 - h) = \cos 0 = 1$$

$$\text{RHL} = \lim_{h\to0} f(0 + h) = \lim_{h\to0}\cos(0 + h)$$

$$= \lim_{h\to0}\cos h = \cos 0 = 1$$

and f(0) = 1

∴ LHL = RHL = f(0)

So, f(x) is continuous at x = 0.

Hence, f(x) is continuous everywhere.

Reason We have, f(x) = cos |x|

$$= \begin{cases} \cos x, & x \geq 0 \\ \cos(-x), & x < 0 \end{cases}$$

$$= \begin{cases} \cos x, & x \geq 0 \\ \cos x, & x < 0 \end{cases}$$

= cos x, x ∈ R

But cos x is always continuous in their domain.
Hence, f(x) is continuous everywhere.
Hence, both Assertion and Reason are true, but Reason is not the correct explanation of Assertion.

40. Sol. **(b)** Both A and R are true but R is not the correct explanation of A.

Explanation: Assertion We have,

$f(x) = \cos(x^2)$

At $x = c$,

$$LHL = \lim_{h\to 0} \cos(c - h)^2 = \cos c^2$$

$$RHL = \lim_{h\to 0} \cos(c + h)^2 = \cos c^2$$

and $f(c) = \cos c^2$

$\therefore$ LHL = RHL = f(c)

So, f(x) is continuous at x = c.

Hence, $f(x)$ is continuous for every value of x.
Hence, both Assertion and Reason are true and Reason is not the correct explanation of Assertion.

41. Sol. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation: Both A and R are true and R is the correct explanation of A.

42. Sol. **(c)** A is true but R is false.

Explanation: Assertion: It is a true statement.

Reason: We have, f(x) = |x|

At x = 0,

$$LHL = \lim_{h\to 0^-} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0^-} \frac{|0-h|-0}{-h}$$

$$= \lim_{h\to 0^-} \frac{h}{-h} = -1$$

$$\text{and } RHL = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{|0+h|-0}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

Here, LHD ≠ RHD, hence f(x) is not continuous at x = 0.

➢ Case study [4 Marks]

43. Ans. Given

$$f(x) = \begin{cases} |x-3|, & x \ge 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4}, & x < 1 \end{cases}$$

$$f(x) = \begin{cases} -(x-3), & 1 \le x < 3 \\ x-3, & x \ge 3 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4}, & x < 1 \end{cases}$$

(i) R.H.D. at x = 1

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{-(1+h-3)-[-(1-3)]}{h}$$

$$= \lim_{h\to 0} \frac{(-h+2-2)}{h} = \lim_{h\to 0} \frac{-h}{h} = -1$$

(ii) L.H.D. at x = 1

$$f'(1) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0} \left[\frac{\left[\frac{(1-h)^2}{4} - \frac{3}{2}(1-h) + \frac{13}{4}\right] - \left[\frac{1}{4} - \frac{3}{2} + \frac{13}{4}\right]}{-h} \right]$$

$$= \lim_{h\to 0} \left[\frac{\frac{1+h^2-2h}{4} - \frac{3}{2} + \frac{3h}{2} + \frac{13}{4} - \frac{1}{4} + \frac{3}{2} - \frac{13}{4}}{-h} \right]$$

$$= \lim_{h\to 0} \left[\frac{\frac{h^2}{4} - \frac{h}{2} + \frac{3h}{2}}{-h} \right] = \lim_{h\to 0} \left[\frac{h^2}{-4h} + \frac{h}{-h} \right] = 0 - 1 = -1$$

(iii) (a) We know, that function f(x) is differentiable at x = 1 if L.H.D. = R.H.D. = f′(1)

$\Rightarrow$ -1 = -1 = -1

Hence, the given function f(x) is differentiable at x = 1.

OR

(b) Differentiate f(x) w.r.t. x, we get

$$f'(x) = \begin{cases} -1, & 1 \le x < 3 \\ 1, & x \ge 3 \\ \dfrac{x}{2} - \dfrac{3}{2}, & x < 1 \end{cases}$$

$$f'(2) = -1 \text{ and } f'(-1) = \frac{-1}{2} - \frac{3}{2} = \frac{-4}{2} = -2$$

➤ Questions

44. Ans. We have, $f(x)=\begin{cases}\dfrac{\sin^2\lambda x}{x^2}, & x\neq 0\\ 1, & x=0\end{cases}$ is continuous at x = 0

$\therefore$ L.H.L. = R.H.L.

$$\Rightarrow \underset{x\to 0}{Lt}\frac{\sin^2\lambda x}{x^2}=1\Rightarrow \underset{x\to 0}{Lt}\frac{\sin^2\lambda x}{\lambda^2x^2}.\lambda^2=1$$

$$\Rightarrow \lambda^2\underset{\lambda x\to 0}{Lt}\left(\frac{\sin\lambda x}{\lambda x}\right)^2=1\Rightarrow \lambda^2.1=1$$

$$\Rightarrow \lambda^2=1\Rightarrow \lambda=\pm 1$$

45. Ans. We have, $y=\sqrt{a+\sqrt{a+x}}$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\frac{1}{\sqrt{a+\sqrt{a+x}}}\times\frac{1}{2}\frac{1}{\sqrt{a+x}}=\frac{1}{4}.\frac{1}{\sqrt{a\sqrt{(a+x)}+a+x}}$$

46. Ans. $\because$ $f(x)$ is continuous at $\pi/2$.

$$\therefore \lim_{x\to\pi/2^-}f(x)=\lim_{x\to\pi/2^+}f(x)=f(\pi/2)\quad ...(1)$$

Now, $\displaystyle\lim_{x\to\pi/2^-}f(x)=\lim_{h\to 0}f\left(\frac{\pi}{2}-h\right)$

$$=\lim_{h\to 0}\frac{1-\sin^3\left(\frac{\pi}{2}-h\right)}{3\cos^2\left(\frac{\pi}{2}-h\right)}=\lim_{h\to 0}\frac{1-\cos^3 h}{3\sin^2 h}$$

$$=\lim_{h\to 0}\frac{(1-\cosh)(1+\cos^2h+\cosh)}{3(1-\cosh)(1+\cosh)}$$

$$=\lim_{h\to 0}\frac{(1+\cos^2h+\cosh)}{3(1+\cosh)}=\frac{1+1+1}{3(1+1)}=\frac{1}{2}$$

And $\displaystyle\lim_{x\to\pi/2^+}f(x)=\lim_{h\to 0}f\left(\frac{\pi}{2}+h\right)$

$$=\lim_{h\to 0}\frac{q\left[1-\sin\left(\frac{\pi}{2}+h\right)\right]}{\left[\pi-2\left(\frac{\pi}{2}+h\right)\right]^2}=\lim_{h\to 0}\frac{q(1-\cosh)}{4h^2}$$

$$=\frac{q}{4}\times\lim_{h\to 0}\frac{2\sin^2\frac{h}{2}}{\frac{h^2}{4}\times 4}=\frac{q}{4}\times\frac{2}{4}=\frac{q}{8}\ and\ f(\pi/2)=p$$

$$\therefore\ \frac{1}{2}=\frac{q}{8}=p$$

$$\Rightarrow\ p=\frac{1}{2}and\ q=4$$

47. Ans. Given that f(x) is differentiable at x = 1.

Therefore, f(x) is continuous at x = 1

$$\Rightarrow \lim_{x\to 1}f(x)=\lim_{x\to 1^+}f(x)=f(1)$$

$$\Rightarrow \lim_{x\to 1}(x^2+3x+a)=\lim_{x\to 1}(bx+2)$$

$$\Rightarrow 1+3+a=b+2\Rightarrow a-b+2=0\quad ....(1)$$

Again, f(x) is differentiable at x = 1. So,

(L.H.D. at x = 1) = (R.H.D. at x = 1)

$$\Rightarrow \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$$

$$\Rightarrow \lim_{x\to 1}\frac{(x^2+3x+a)-(4+a)}{x-1}=\lim_{x\to 1}\frac{(bx+2)-(4+a)}{x-1}$$

$$\Rightarrow \lim_{x\to}\frac{x^2+3x-4}{x-1}=\lim_{x\to 1}\frac{bx-2-a}{x-1}$$

$$\Rightarrow \lim_{x\to}\frac{(x+4)(x-1)}{x-1}=\lim_{x\to 1}\frac{bx-b}{x-1}\qquad \text{[From (1)]}$$

$$\Rightarrow \lim_{x\to 1}(x+4)=\lim_{x\to 1}\frac{b(x-1)}{x-1}\Rightarrow 5=b$$

Putting b = 5 in (1), we get a = 3.

Hence, a = 3 and b = 5

48. Ans. We have, $y=e^{x^2\cos x}+(\cos x)^x$

$$=e^{x^2\cos x}+e^{x(\ln\cos x)}$$

$$\therefore \frac{dy}{dx}=e^{x^2\cos x}\frac{d}{dx}(x^2\cos x)+e^{x\ln\cos x}\frac{d}{dx}(x\ln\cos x)$$

$$=e^{x^2\cos x}(2x\cos x-x^2\sin x)$$

$$+e^{x\ln\cos}\left(\ln\ \cos x-\frac{x}{\cos x}\sin x\right)$$

$$=e^{x^2\cos x}(2x\cos x-x^2\sin x)+(\cos x)^x(\ln\cos x-x\tan x)$$

49. Ans. Given, $\log(x^2+y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$

On differentiating w.r.t. x on both sides, we get

$$\frac{1}{x^2+y^2}\left(2x+2y\frac{dy}{dx}\right) = 2\times\frac{1}{1+\frac{y^2}{x^2}}\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{2x}{x^2+y^2}+\frac{2y}{x^2+y^2}\frac{dy}{dx} = \frac{2x^2}{x^2+y^2}\left(\frac{1}{x}\frac{dy}{dx}+y\left(\frac{-1}{x^2}\right)\right)$$

$$\Rightarrow \frac{dy}{dx}\left[\frac{2y}{x^2+y^2}-\frac{2x}{x^2+y^2}\right] = \frac{2x^2}{x^2+y^2}\left[\frac{-y}{x^2}-\frac{1}{x}\right]$$

$$\Rightarrow \frac{2(y-x)}{x^2+y^2}\frac{dy}{dx} = \frac{-2x^2}{x^2+y^2}\left(\frac{y+x}{x^2}\right) \Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y}$$

50. Ans. Given $y = (x)^{\cos x} + (\cos x)^{\sin x}$

Let $u = (x)^{\cos x}$, $v = (\cos x)^{\sin x}$ $\therefore y = u+v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx}+\frac{dv}{dx} \quad \text{.......(i)}$$

Now, $u = x^{\cos x}$

$\Rightarrow \log u = \cos x \log x$

Differentiating with respect to x, we get

$$\frac{1}{u}\frac{du}{dx} = \cos x.\frac{1}{x}+\log x(-\sin x)$$

$$\Rightarrow \frac{du}{dx} = x^{\cos x}\left[\frac{\cos x}{x}-\sin x\log x\right] \text{.....(ii)}$$

Now, $v = (\cos x)^{\sin x}$

$\Rightarrow \log v = \sin x \log\cos x$

Differentiating with respect to x, we get

$$\frac{1}{v}\frac{dv}{dx} = \sin x.\frac{1}{\cos x}(-\sin x)+\log\cos x.\cos x$$

$$\Rightarrow \frac{dv}{dx} = (\cos x)^{\sin x}\left[\frac{-\sin^2 x+\cos^2 x\log\cos x}{\cos x}\right]$$

..........(iii)

Putting value of (ii) and (iii) into (i), we get

$$\frac{dy}{dx} = x^{\cos x}\left[\frac{\cos x}{x}-\sin x\log x\right] + (\cos x)^{\sin x}\left[\frac{-\sin^2 x+\cos^2 x\log\cos x}{\cos x}\right]$$

51. Ans. Let $u = \tan^{-1}\left[\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right]$ ....(i)

Putting $x^2 = \cos 2\theta$ in (i), we get

$$u = \tan^{-1}\left[\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right]$$

$$= \tan^{-1}\left[\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right] = \tan^{-1}\left[\frac{1-\tan\theta}{1+\tan\theta}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\theta\right)\right] = \frac{\pi}{4}-\theta$$

$$= \frac{\pi}{4}-\frac{\cos^{-1}(x^2)}{2}$$

$$\left(\because x^2 = \cos 2\theta \Rightarrow \theta = \frac{\cos^{-1}x^2}{2}\right)$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{2}\frac{1}{\sqrt{1-x^4}}2x = \frac{x}{\sqrt{1-x^4}} \quad \text{..........(ii)}$$

Let $v = \cos^{-1}(x^2) \Rightarrow \frac{dv}{dx} = \frac{-2x}{\sqrt{1-x^4}}$ ....(iii)

Dividing (ii) by (iii), we get $\frac{du}{dv} = -\frac{1}{2}$.

52. Ans. Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$\therefore u = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) \Rightarrow u = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) \Rightarrow u = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$\therefore u = \frac{\theta}{2} \Rightarrow u = \frac{1}{2}\tan^{-1}x$$

$$\left(\because 2\sin^2\frac{x}{2} = 1-\cos x, 2\sin\frac{x}{2}\cos\frac{x}{2} = \sin x\right)$$

Differentiating w.r.t. x, we get $\frac{du}{dx} = \frac{1}{2(1+x^2)}$

Also, let $v=\sin^{-1}\left(\frac{2x}{1+x^2}\right)\Rightarrow v=2\tan^{-1}x$

$$\frac{dv}{dx}=\frac{2}{1+x^2}\therefore\ \frac{du}{dv}=\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{\frac{1}{2(1+x^2)}}{\frac{2}{1+x^2}}\Rightarrow\frac{du}{dv}=\frac{1}{4}$$

53. Soln. Since f(x) is continuous at x = 3, we have:

$$\lim_{x\to3^{-1}}f(x)=\lim_{x\to3^+}f(x)=f(3)\qquad .....(i)$$

Now, $\lim_{x\to3}f(x)=\lim_{x\to3^{-}}(ax+1)$

$\lim_{h\to0}[a(3-h)+1]=3a+1$

Similarly,

$$\lim_{x\to3^+}(bx+3)=\lim_{h\to0}[b(3+h)+3]$$
$$=3b+3$$

From (i), we can equate:

$$3a+1=3b+3$$
$$\Rightarrow\quad 3(a-b)=2$$
$$or\quad a-b=\frac{2}{3}$$

54. Soln. Since f(x) is continuous at x = 2, we have

$$\lim_{x\to2^-}f(x)=\lim_{x\to2^+}f(x)=f(2)\qquad .....(i)$$

Now $\lim_{x\to2^-}f(x)=\lim_{x\to2^-}(2x-1)$

$$=\lim_{h\to0}[2(2-h)-1]=3$$

Also, we are given

$$f(2)=a$$

Therefore, from (i), we can write:

$$f(2)=3\text{ or }a=3.$$

55. Soln. Since f(x) is continuous at x = 0, therefore

$$\lim_{x\to0^-}f(x)=\lim_{x\to0^+}f(x)=f(0)$$

$$\Rightarrow\ \lim_{h\to0}f(0-h)=\lim_{h\to0}f(0+h)=k$$

$$\Rightarrow\ \lim_{h\to0}\frac{\sin(-2h)}{-5h}=\lim_{h\to0}\frac{\sin2h}{5h}=k$$

$$\Rightarrow\ \lim_{h\to0}\frac{\sin2h}{2h}\times\frac{2}{5}=k$$

$$\Rightarrow\quad k=\frac{2}{5}$$

56. Soln.

Given, f(x) = $|x-3|,\ x\in R$

$$=\begin{cases}x-3, & if\ x\ge3\\ 3-x, & if\ x<3\end{cases}$$

When x > 3, f(x) = x – 3 and it is a polynomial, so it is continuous.

When x < 3, f(x) = 3 – x. Again it is a polynomial, so it is continuous.

Also $f(3-0)=0=f(3+0)=f(3)$

$\Rightarrow$ $f(x)$ is continuous at x = 3.

Now, LHD = $f'(3-0)$ = $\lim_{x\to3-h}\frac{f(x)-f(3)}{-h}$

$$=\lim_{h\to0}\frac{3-(3-h)-0}{-h}$$

$$=\lim_{h\to0}\frac{h}{-h}=-1.$$

And RHD = $f'(3+0)$ = $\lim_{x\to3+h}\frac{f(x)-f(3)}{h}$

$$=\lim_{h\to0}\frac{(3+h)-3-0}{h}$$

$$=\lim_{h\to0}\frac{h}{h}=1.$$

Since $\quad LHD\ne RHD$

$\Rightarrow$ f is not differentiable at x = 3. Hence Proved

57. Soln. If f is a continuous function, f is continuous at all real numbers.

In particular, f(x) is continuous at x = 2 and x = 10.

Since f is continuous at x = 2, we obtain

$$\lim_{x\to2^-}f(x)=\lim_{x\to2^+}f(x)=f(2)$$

$$\Rightarrow \lim_{x\to 2^-}(5)=\lim_{x\to 2^+}(ax+b)=5$$

$$\Rightarrow \quad 5=2a+b=5$$

$$\Rightarrow \quad 2a+b=5 \quad \text{........(i)}$$

Since f is continuous at x = 10, we obtain

$$\lim_{x\to 10^-} f(x)=\lim_{x\to 10^+} f(x)=f(10)$$

$$\Rightarrow \lim_{x\to 10^-}(ax+b)=\lim_{x\to 10^+}(21)=21$$

$$\Rightarrow \quad 10a+b=21 \quad \text{.........(ii)}$$

On subtracting equation (i) from equation (ii), we obtain

8a = 16

$$\Rightarrow \quad a=2$$

By putting a = 2 in equation (i), we obtain

$$2\times 2+b=5$$

$$\Rightarrow \quad 4+b=5$$

$$\Rightarrow \quad b=1$$

Therefore, the values of a and b for which f(x) is a continuous function are 2 and 1 respectively.

58. Soln. Given, that the function is continuous at x = 0

$$\therefore \lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0) \quad \text{.......(i)}$$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{kx}{|x|}$$

$$=k\lim_{x\to 0^-}\frac{x}{-x}$$

$$=k(-1)=-k$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}3=3$$

From equation (i),

-k = 3

Or k = -3

59. Soln. We have

$$f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x}, & if \ x<\dfrac{\pi}{2}\\ p, & if \ x=\dfrac{\pi}{2}\\ \dfrac{q(1-\sin x)}{(\pi-2x)^2}, & if \ x>\dfrac{\pi}{2}\end{cases}$$ is continuous at

$$x=\frac{\pi}{2}$$

Now $\lim_{h\to\frac{\pi}{2}^+} f(x)=\lim_{h\to 0} f\left(\frac{\pi}{2}+h\right)$

$$\left[Let\ x=\frac{\pi}{2}+h, x\to\frac{\pi}{2}^+\Rightarrow h\to 0\right]$$

$$=\lim_{h\to 0}\frac{q\left\{1-\sin\left(\frac{\pi}{2}+h\right)\right\}}{\left\{\pi-2\left(\frac{\pi}{2}+h\right)\right\}^2}=\lim_{h\to 0}\frac{q[1-\cos h]}{\{\pi-\pi-2h\}^2}=\lim_{h\to 0}\frac{q(1-\cos h)}{4h^2}$$

$$=\lim_{h\to 0}\frac{q.2\sin^2\frac{h}{2}}{4h^2}=\lim_{h\to 0}\frac{q.\sin^2\frac{h}{2}}{2h^2}$$

$$=q.\lim_{h\to 0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2\times\frac{1}{8}=\frac{q}{8}$$

Again $\lim_{h\to\frac{\pi}{2}} f(x)=\lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)$

$$\left[Let\ x=\frac{\pi}{2}-h, x\to\frac{\pi}{2}^-\Rightarrow h\to 0\right]$$

$$=\lim_{h\to 0}\frac{1-\sin^3\left(\frac{\pi}{2}-h\right)}{3\cos^2\left(\frac{\pi}{2}-h\right)}=\lim_{h\to 0}\frac{1-\cos^3 h}{3\sin^2 h}$$

$$=\lim_{h\to 0}\frac{(1-\cos h)(1+\cos h+\cos^2 h)}{3\sin^2 h}$$

$$=\lim_{h\to 0}\frac{2\sin^2\frac{h}{2}.(1+1+1)}{3\sin^2 h}$$

$$=\lim_{h\to 0}\frac{2\sin^2\frac{h}{2}.3}{3\sin^2 h}=\lim_{h\to 0}\frac{2.\sin^2\frac{h}{2}}{\sin^2 h}$$

$$=\lim_{h\to 0}\frac{2.\frac{\sin^2\frac{h}{2}}{h^2}}{\frac{\sin^2 h}{h^2}}$$ [Dividing N^r and D^r by h^2]

$$=\lim_{h\to 0}\frac{2.\frac{\sin^2\frac{h}{2}}{\frac{h^2}{4}\times 4}}{\frac{\sin^2 h}{h^2}}=\frac{1}{2}\frac{\left(\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2}{\left(\lim_{h\to 0}\frac{\sin h}{h}\right)^2}=\frac{1}{2}$$

Also $f\left(\frac{\pi}{2}\right)=p$

$\because$ $f(x)$ is continuous at $x=\frac{\pi}{2}$

$\because$ $\lim\limits_{h\to\frac{\pi^+}{2}} f(x)=\lim\limits_{h\to\frac{\pi^-}{2}} f(x)=f\left(\frac{\pi}{2}\right)$

$\Rightarrow$ $\frac{q}{8}=\frac{1}{2}=p$

$\Rightarrow$ $p=\frac{1}{2}$ *and* $q=4$

60. Soln. Here f(x) = 2x - |x|

For continuity at x = 0

$$\lim_{x\to0^+} f(x)=\lim_{h\to0} f(0+h)=\lim_{h\to0} f(h)$$

$$=\lim_{h\to0}\{2h-|h|\}=\lim_{h\to0}(2h-h)$$

$$=\lim_{h\to0} h$$

$$=0 \qquad \text{........(i)}$$

$$\lim_{x\to0^-} f(x)=\lim_{h\to0} f(0-h)=\lim_{h\to0} f(-h)$$

$$=\lim_{h\to0}\{2(-h)-|-h|\}=\lim_{h\to0}\{-2h-h\}$$

$$=\lim_{h\to0}(-3h)$$

$$=0 \qquad \text{.......(ii)}$$

$$=\text{Also } f(0)=2\times0-|0|=0 \qquad \text{..........(iii)}$$

$$\text{(i), (ii) and (iii)} \Rightarrow \lim_{x\to0^+} f(x)=\lim_{x\to0^-} f(x)=f(0)$$

Hence, f(x) is continuous at x = 0

For differentiability at x = 0

$$LHD=\lim_{h\to0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to0}\frac{f(-h)-f(0)}{-h}$$

$$=\lim_{h\to0}\frac{(2(-h)-|-h|)-\{2\times0-|0|\}}{-h}=\lim_{h\to0}\frac{-2h-h-0}{-h}$$

$$=\lim_{h\to0}\frac{-3h}{-h}=\lim_{h\to0}3$$

$$LHD \;=3 \qquad \text{...........(iv)}$$

Again $RHD=\lim\limits_{h\to0}\frac{f(0+h)-f(0)}{h}$

$$=\lim_{h\to0}\frac{f(h)-f(0)}{h}=\lim_{h\to0}\frac{2h-|h|-2\times0-|0|}{h}$$

$$=\lim_{h\to0}\frac{2h-h}{h}=\lim_{h\to0}\frac{h}{h}$$

$$=\lim_{h\to0}1.$$

$$RHD=1 \qquad \text{...........(v)}$$

From (iv) and (v)

$LHD\neq RHD$

i.e., function f(x) = 2x - |x| is not differentiable at x = 0

Hence, f(x) is continuous but not differentiable at x = 0.

61. Soln.

$\because$ $f(x)$ is continuous at x = 0.

$\Rightarrow$ (LHL of f(x) at x = 0) = (RHL of f(x) at x = 0) f(0)

$$\Rightarrow \lim_{x\to0^-} f(x)=\lim_{x\to0^+} f(x)=f(0)$$

.........(i)

$\lim\limits_{x\to0^-} f(x)=\lim\limits_{h\to0} f(0-h)$ [Let x = 0 – h, $x\to0^-\Rightarrow h\to0$]

$$=\lim_{h\to0} f(-h)=\lim_{h\to0}\frac{\sqrt{1+k(-h)}-\sqrt{1-k(-h)}}{-h}$$

$$=\lim_{h\to0}\frac{\sqrt{1-kh}-\sqrt{1+kh}}{-h}\times\frac{\sqrt{1-kh}+\sqrt{1+kh}}{\sqrt{1-kh}+\sqrt{1+kh}}$$

$$=\lim_{h\to0}\frac{(1-kh)-(1+kh)}{-h\{\sqrt{1-kh}+\sqrt{1+kh}\}}=\lim_{h\to0}\frac{2k}{\{\sqrt{1-kh}+\sqrt{1+kh}\}}=\frac{2k}{2}$$

$$\Rightarrow \lim_{x\to0^-} f(x)=k \qquad \text{............(ii)}$$

Again $\lim\limits_{x\to0^+} f(x)=\lim\limits_{h\to0} f(0+h)$

[Let x = 0 + h, $x\to0^+\Rightarrow h\to0$]

$$=\lim_{h\to0} f(h)=\lim_{h\to0}\frac{2h+1}{h-1}=\frac{1}{-1}$$

$$\Rightarrow \lim_{x\to0^+} f(x)=-1 \qquad \text{.........(iii)}$$

Also $f(0)=\frac{2\times0+1}{0-1}=-1$ ....(iv)

$\because$ f(x) is continuous at x = 0

$\therefore$ (i), (ii), (iii) and (iv) $\Rightarrow$ $k=-1$.

62. Soln. $\because$ f(x) is continuous at x = 0

$\Rightarrow$ (LHL of f(x) at x = 0) = (RHL of f(x) at x = 0) = f(0)

$$\Rightarrow \lim_{x\to0^-} f(x)=\lim_{x\to0^-} f(x)=f(0) \qquad \text{........(i)}$$

Now, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0} a\sin\frac{\pi}{2}(x+1)$

$$\left[\because f(x) = a\sin\frac{\pi}{2}(x+1), \text{if } x \le 0\right]$$

$$= \lim_{x\to 0} a\sin\left(\frac{\pi}{2}+\frac{\pi}{2}x\right) = \lim_{x\to 0} a\cos\frac{\pi}{2}x = a.\cos 0 = a$$

.........(ii)

Again, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0}\frac{\tan x - \sin x}{x^3}$

$$\left[\because f(x) = \frac{\tan x - \sin x}{x^3}\ if\ x > 0\right]$$

$$= \lim_{x\to 0}\frac{\frac{\sin x}{\cos x} - \sin x}{x^3} = \lim_{x\to 0}\frac{\sin x - \sin x.\cos x}{\cos x.x^3}$$

$$= \lim_{x\to 0}\frac{\sin x(1-\cos x)}{\cos x.x^3}$$

$$= \lim_{x\to 0}\frac{1}{\cos x}.\lim_{x\to 0}\frac{\sin x}{x}.\frac{2\sin^2\frac{x}{2}}{\frac{x^2}{4}\times 4}$$

$$\left[\because 1-\cos x = 2\sin^2\frac{x}{2}\right]$$

$$= \frac{1}{1}.1.\frac{1}{2}\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2}.\left(\lim_{\frac{x}{2}\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2}\times 1 = \frac{1}{2}$$

........(iii)

Also, $f(0) = a\sin\frac{\pi}{2}(0+1) = a\sin\frac{\pi}{2} = a$ .....(iv)

$\because$ f is continuous at x = 0

$\therefore$ (i), (ii), (iii) and (iv) $\Rightarrow$ $a = \frac{1}{2}$

$$f(x) = \begin{cases}\frac{x}{2|x|}, & x \ne 0\\ \frac{1}{2} & ,x = 0\end{cases}$$

63. Soln. Given,

For continuity at x = 0, we have

$$f(0) = \frac{1}{2}$$

LHL = $\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{x}{2|x|} = \lim_{x\to 0}\frac{x}{-2x} = -\frac{1}{2}$

RHL = $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{x}{2|x|} = \lim_{x\to 0^+}\frac{x}{2x} = \frac{1}{2}$

Hence, $LHL \ne RHL$

So, f(x) is discontinuous at x = 0.

$$f(x) = \begin{cases}\frac{k\cos x}{\pi - 2x}, & if\ \ x \ne \frac{\pi}{2}\\ 3, & if\ \ x = \frac{\pi}{2}\end{cases}$$

64. Soln. We have,

For f(x) to be continuous at x = 0,

$$LHL = RHL = f\left(\frac{\pi}{2}\right)$$

Now, $f\left(\frac{\pi}{2}\right) = 3$ .....(i)

LHL $= \lim_{x\to\frac{\pi^-}{2}} f(x) = \lim_{x\to\frac{\pi^-}{2}}\frac{k\cos x}{\pi - 2x}$

$$= \lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}-h\right)}{\pi - 2\left(\frac{\pi}{2}-h\right)}$$ [We assume $x = \frac{\pi}{2} - h$ so that $x \to \frac{\pi}{2}$, $h \to 0$]

$$= \lim_{h\to 0}\frac{k\sin h}{2h} = \lim_{h\to 0}\frac{k}{2}\times\left(\frac{\sin h}{h}\right) = \frac{k}{2}$$

$$\left(\because \lim_{h\to 0}\frac{\sin h}{h} = 1\right)$$ ....(ii)

RHL = $\lim_{x\to\frac{\pi^+}{2}} f(x) = \lim_{x\to\frac{\pi^+}{2}}\frac{k\cos x}{\pi - 2x}$

$$= \lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi - 2\left(\frac{\pi}{2}+h\right)}$$

$$\left[Let\ x = \frac{\pi}{2}+h,\ h\to 0\ as\ x \to \frac{\pi^+}{2}\right]$$

$$= \lim_{h\to 0}\frac{-k\sin h}{-2h} = \lim_{h\to 0}\frac{k}{2}\times\frac{\sin h}{h} = \frac{k}{2}$$ .....(iii)

Hence, (i), (ii) and (iii) $\Rightarrow$ $\frac{k}{2} = 3$ or $k = 6$

65. Soln. For continuity:

$$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2-h)$$

$$[\text{Let } x = 2-h \Rightarrow x \to 2^- \Rightarrow h \to 0]$$

$$= \lim_{h\to 0} 2(2-h)^2 - (2-h)$$

$$= \lim_{h\to 0} 2\{4 + h^2 - 4h\} - (2-h)$$

$$= \lim_{h\to 0} (8 + 2h^2 - 8h - 2 + h) = 6$$

$$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h)$$

$$[\text{Let } x = 2 + h \Rightarrow x \to 2^+ \Rightarrow h \to 0]$$

$$= \lim_{h\to 0} 5(2+h) - 4 = 6$$

$$f(2) = 2.2^2 - 2 = 6$$

i.e., $\lim_{x\to 2} f(x) = \lim_{x\to 2^+} f(x) = f(2)$

$\Rightarrow f(x)$ is continuous at x = 2.

For Differentiability:

LHD (at x = 2) = $\lim_{h\to 0} \frac{f(2-h) - f(2)}{-h}$

$$= \lim_{h\to 0} \frac{2(2-h)^2 - (2-h) - \{2.2^2 - 2\}}{-h}$$

$$= \lim_{h\to 0} \frac{8 + 2h^2 - 8h - 2 + h - 6}{-h}$$

$$= \lim_{h\to 0} \frac{2h^2 - 7h}{-h} = \lim_{h\to 0} \frac{2h-7}{-1} = $$

RHD (at x = 2) = $\lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$

$$= \lim_{h\to 0} \frac{5(2+h) - 4 - \{2.2^2 - 2\}}{h}$$

$$= \lim_{h\to 0} \frac{10 + 5h - 4 - 6}{h} = \lim_{h\to 0} 5 = 5$$

$\because\ LHD \neq RHD$ (at x = 2)

Hence, f(x) is not differentiable at x = 2.

66. Soln. Here, given function is

$$f(x) = |x-1| + |x+1|$$

$$\Rightarrow f(x) = \begin{cases} -(x-1)-(x+1), & x < -1 \\ 2, & x = -1 \\ -(x-1)+(x+1), & -1 < x < 1 \\ 2 & x = 1 \\ (x-1)+(x+1) & x > 1 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} -2x, & \text{if } x < -1 \\ 2, & \text{if } x = -1 \\ 2, & \text{if } -1 < x < 1 \\ 2, & \text{if } x = 1 \\ 2x, & \text{if } x > 1 \end{cases} \Rightarrow f(x) = \begin{cases} -2x & \text{if } x < -1 \\ 2 & \text{if } -1 \le x \le 1 \\ 2x & \text{if } x > 1 \end{cases}$$

For x = -1

Now, RHD = $\lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h}$

$$= \lim_{h\to 0} \frac{2-2}{h} = \lim_{h\to 0} 0 = 0$$

LHD = $\lim_{h\to 0} \frac{f(-1-h) - f(-1)}{-h}$

$$= \lim_{h\to 0} \frac{-2(-1-h) - 2}{-h} = \lim_{h\to 0} \frac{2 + 2h - 2}{-h}$$

$$= \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} -2 = -2.$$

i.e., $RHD \neq LHD$.

Hence, f(x) is not differentiable at x = -1.

For x = 1

RHD $= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$

$$= \lim_{h\to 0} \frac{2(1+h) - 2}{h} = \lim_{h\to 0} \frac{2 + 2h - 2}{h}$$

$$= \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} 2 = 2$$

LHD $= \lim_{h\to 0} \frac{f(1-h) - f(1)}{-h}$

$$= \lim_{h\to 0} \frac{2-2}{-h} = \lim_{h\to 0} \frac{0}{-h}$$

$$= \lim_{h\to 0} 0 = 0$$

$RHD \neq LHD$.

Hence, f(x) not differentiable at x = 1.

67. Soln. Given, $x^y = e^{x-y}$

Taking log both sides, we get

$$\Rightarrow \log x^y = \log e^{x-y}$$

$$\Rightarrow y.\log x = (x-y)\log e \quad [\because \log e = 1]$$

$$\Rightarrow y = \frac{x}{1 + \log x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+\log x).1 - x.\left(0 + \frac{1}{x}\right)}{(1+\log x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 + \log x - 1}{(1+\log x)^2} = \frac{\log x}{(\log e + \log x)^2} \quad [\because 1 = \log e]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\log x}{(\log ex)^2} \Rightarrow \frac{dy}{dx} = \frac{\log x}{\{\log(ex)\}^2}.$$

68. Soln. Given, $(\cos x)^y = (\cos y)^x$

Taking logarithm both sides, we get

$$\log(\cos x)^y = \log(\cos y)^x$$

$$\Rightarrow y.\log(\cos x) = x.\log(\cos y)$$

$[\because \log m^n = n\log m]$

Differentiating both sides, we get

$$y.\frac{1}{\cos x}(-\sin x) + \log(\cos x).\frac{dy}{dx}$$

$$= x.\frac{1}{\cos y}.(-\sin y).\frac{dy}{dx} + \log(\cos y)$$

$$\Rightarrow -\frac{y\sin x}{\cos x} + \log(\cos x).\frac{dy}{dx} = -\frac{x\sin y}{\cos y}.\frac{dy}{dx} + \log(\cos y)$$

$$\Rightarrow \log(\cos x).\frac{dy}{dx} + \frac{x\sin y}{\cos y}.\frac{dy}{dx} = \log(\cos y) + \frac{y\sin x}{\cos x}$$

$$\Rightarrow \frac{dy}{dx}\left[\log(\cos x) + \frac{x\sin y}{\cos y}\right] = \log(\cos y) + \frac{y\sin x}{\cos x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\log(\cos y) + \frac{y\sin x}{\cos x}}{\log(\cos x) + \frac{x\sin y}{\cos y}} = \frac{\log(\cos y) + y\tan x}{\log(\cos x) + x\tan y}$$

69. Soln.

Consider $\dfrac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}$

$$= \frac{\sqrt{1+2\sin\frac{x}{2}\cos\frac{x}{2}} + \sqrt{1-2\sin\frac{x}{2}\cos\frac{x}{2}}}{\sqrt{1+2\sin\frac{x}{2}\cos\frac{x}{2}} - \sqrt{1-2\sin\frac{x}{2}\cos\frac{x}{2}}}$$

$$\left[\because \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right]$$

$$= \frac{\sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}.\cos\frac{x}{2}} + \sqrt{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}}{\sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}} - \sqrt{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}}$$

$$= \frac{\cos\frac{x}{2} + \sin\frac{x}{2} + \cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2} - \cos\frac{x}{2} + \sin\frac{x}{2}} = \frac{2\cos\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)} = \cot\frac{x}{2}$$

$$\therefore y = \cot^{-1}\left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right]$$

$$= \cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}.$$

70. Soln. Given, $x = a\sin 2t(1+\cos 2t)$ and

$y = b\cos 2t(1-\cos 2t)$

$$\Rightarrow \frac{dx}{dt} = a[\sin 2t \times (-2\sin 2t) + (1+\cos 2t)\times 2\cos 2t]$$

$$= a[-2\sin^2 2t + 2\cos 2t + 2\cos^2 2t]$$

$$= a(2\cos 4t + 2\cos 2t) = 2a(\cos 4t + \cos 2t)$$

Again,

$$\frac{dy}{dt} = b[\cos 2t \times 2\sin 2t + (1-\cos 2t)\times -2\sin 2t]$$

$$= b[\sin 4t - 2\sin 2t + \sin 4t] = b[2\sin 4t - 2\sin 2t]$$

$$= 2b(\sin 4t - \sin 2t)$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2b(\sin 4t - \sin 2t)}{2a(\cos 4t + \cos 2t)}$$

$$= \frac{b}{a}\left[\frac{\sin 4t - \sin 2t}{\cos 4t + \cos 2t}\right]$$

$$\therefore \left(\frac{dy}{dx}\right)_{at\ t=\frac{\pi}{4}} = \frac{b}{a}\left(\frac{\sin\pi - \sin\frac{\pi}{2}}{\cos\pi + \cos\frac{\pi}{2}}\right) = \frac{b}{a}\times\frac{-1}{-1}$$

Hence, $\dfrac{dy}{dx}_{at\ t=\frac{\pi}{4}} = \dfrac{b}{a}$

$$\left(\frac{dy}{dx}\right)_{at\ t=\frac{\pi}{3}} = \frac{b}{a}\left(\frac{\sin\frac{4\pi}{3} - \sin\frac{2\pi}{2}}{\cos\frac{4\pi}{3} + \cos\frac{2\pi}{3}}\right)$$

$$= \frac{b}{a}\left(\frac{-\sin\frac{\pi}{3} - \sin\frac{\pi}{3}}{-\cos\frac{\pi}{3} - \cos\frac{\pi}{3}}\right) = \frac{b}{a}\times\frac{-2\sin\frac{\pi}{3}}{-2\cos\frac{\pi}{3}} = \frac{b}{a}\tan\frac{\pi}{3} = \frac{\sqrt{3}b}{a}$$

71. Soln. Let

$y = x^{\sin x} + (\sin x)^{\cos x}, u = x^{\sin x}, v = (\sin x)^{\cos x}$

Now, $y = u + v$

And $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ ....(i)

We have, $u = x^{\sin x}$

Taking log on both sides, we get

$$\log u = \sin x \log x$$

Differentiating both sides with respect to x, we get

$$\frac{1}{u}\frac{du}{dx} = \sin x \times \frac{1}{x} + \log x.\cos x$$

$$\frac{du}{dx} = u\left(\frac{\sin x}{x} + \log x.\cos x\right)$$

$$= x^{\sin x}\left[\frac{\sin x}{x} + \log x.\cos x\right] \quad ......(ii)$$

Again, $v = (\sin x)^{\cos x}$

Taking log on both sides with respect to x, we get

$$\log v = \cos x \log \sin x$$

Differentiating both sides with respect to x, we get

$$\frac{1}{v}\frac{dv}{dx} = \cos x \times \frac{1}{\sin x} \times \cos x + \log \sin x(-\sin x)$$

$$= \cos x.\cot x + \log \sin x.(-\sin x)$$

$$= \cos x.\cot x - \sin x.\log \sin x$$

$$\therefore \frac{dv}{dx} = v[\cos x.\cot x - \sin x.\log \sin x]$$

$$\frac{dv}{dx} = (\sin x)^{\cos x}[\cos x.\cot x - \sin x.\log \sin x]$$

...........(iii)

From (i), (ii) and (iii), we get

$$\frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \log x.\cos x\right] + (\sin x)^{\cos x}[\cos x \cot x - \sin x \log \sin x]$$

72. Soln. Given, $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

$y = u + v$, where $u = (\sin x)^x, v = \sin^{-1}\sqrt{x}$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ....(i)$$

Now, $u = (\sin x)^x$

Taking log both sides, we get

$$\log u = \log(\sin x)^x \Rightarrow \log u = x.\log(\sin x)$$

Differentiating both sides with respect to x we get

$$\frac{1}{u}.\frac{du}{dx} = x.\frac{1}{\sin x}\cos x + \log \sin x$$

$$\Rightarrow \frac{du}{dx} = u\{x\cot x + \log \sin x\}$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x\{x\cot x + \log \sin x\} \quad ....(ii)$$

Also, $v = \sin^{-1}\sqrt{x}$

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \times \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(1-x)}}$$

.........(iii)

From (i), (ii) and (iii), we get

$$\therefore \frac{dy}{dx} = (\sin x)^x\{x\cot x + \log \sin x\} + \frac{1}{2\sqrt{x(1-x)}}$$

73. Given, $x = a\cos\theta + b\sin\theta$

$$\Rightarrow \frac{dx}{d\theta} = -a\sin\theta + b\cos\theta \quad ....(i)$$

Also, $y = a\sin\theta - b\cos\theta$

$$\frac{dy}{d\theta} = a\cos\theta + b\sin\theta \quad ....(ii)$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\cos\theta + b\sin\theta}{-a\sin\theta + b\cos\theta}$$ [From (i) and (ii)]

$$\Rightarrow \frac{dy}{dx} = \frac{a\cos\theta + b\sin\theta}{b\cos\theta - a\sin\theta}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y} \quad ....(iii)$$

Differentiating again with respect to x, we get

$$\frac{d^2y}{dx^2} = -\frac{y - x.\frac{dy}{dx}}{y^2}$$

$$\Rightarrow \quad y^2\frac{d^2y}{dx^2} = -y + x\frac{dy}{dx}$$

$$\Rightarrow \quad y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

74. Soln. Given, $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$

Differentiating with respect to t, we get

$$\frac{dx}{dt}=a\left(-\sin t+\frac{1}{\tan\frac{t}{2}}.\sec^2\frac{t}{2}.\frac{1}{2}\right)$$

$$=a\left\{-\sin t+\frac{1}{2\sin\frac{t}{2}.\cos\frac{t}{2}}\right\}=a\left\{-\sin t+\frac{1}{\sin t}\right\}$$

$$\frac{dx}{dt}=a\left(\frac{1-\sin^2 t}{\sin t}\right)=a\frac{\cos^2 t}{\sin t}$$

$\because \quad y=a\sin t$

Differentiating with respect to t, we get

$$\frac{dy}{dt}=a.\cos t\Rightarrow\frac{d^2y}{dt^2}=-a\sin t$$

$$\therefore\ \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{a\cos t.\sin t}{a\cos^2 t}=\tan t$$

$$\therefore\ \frac{d^2y}{dx^2}=\sec^2 t.\frac{dt}{dx}=\sec^2 t.\frac{1\times\sin t}{a\cos^2 t}=\frac{1}{a}\sec^4 t.\sin t$$

Hence, $\dfrac{d^2y}{dt^2}=-a\sin t$ *and* $\dfrac{d^2y}{dx^2}=\dfrac{\sec^4 t\sin t}{a}$

75. Soln. Given, $y=e^{x+e^{x+e^{x+\ldots\text{to}\infty}}}$

$\therefore \quad y=e^{x+y}$ ....(i)

Taking logarithm, we get $\log y=(x+y)\log_e e$

Or $\log y=x+y$ $\quad[\because \log_e e=1]$

Differentiating w.r.t. x, we get

$$\frac{1}{y}.\frac{dy}{dx}=1+\frac{dy}{dx}\Rightarrow\left(\frac{1}{y}-1\right)\frac{dy}{dx}=1$$

$$\Rightarrow\left(\frac{1-y}{y}\right)\frac{dy}{dx}=1$$

$$\therefore\ \frac{dy}{dx}=\frac{y}{1-y}$$

76. Soln.

Let $y=\tan^{-1}\left(\dfrac{\sqrt{x}-x}{1+x^{3/2}}\right)$

Then,

$$y=\tan^{-1}\left(\frac{\sqrt{x}-x}{1+\sqrt{x}.x}\right)=\tan^{-1}\left(\sqrt{x}\right)-\tan^{-1}(x)$$

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}(\tan^{-1}\sqrt{x})-\frac{d}{dx}(\tan^{-1}x)$$

$$=\frac{d(\tan^{-1}\sqrt{x})}{d\sqrt{x}}.\frac{d\sqrt{x}}{dx}-\frac{1}{1+x^2}$$

$$=\frac{1}{1+x}.\frac{1}{2\sqrt{x}}-\frac{1}{1+x^2}$$

$$=\frac{1}{2\sqrt{x}(1+x)}-\frac{1}{1+x^2}$$

77. Soln. Here, $y=Ae^{-kt}\cos(pt+c)$

Differentiating w.r.t. t, we get

$$\frac{dy}{dt}=-k\,Ae^{-kt}\cos(pt+c)-pAe^{-kt}\sin(pt+c)$$

Or $\dfrac{dy}{dt}=-ky-pAe^{-kt}\sin(pt+c)$ ....(i)

Differentiating (i) again w.r.t. 't', we get

$$\frac{d^2y}{dt^2}=-k\frac{dy}{dt}+pkAe^{-kt}\sin(pt+c)-p^2Ae^{-kt}\cos(pt+c)$$

$$=-\frac{k\,dy}{dt}+pk\ Ae^{-kt}\sin(pt+c)-p^2y$$

$$\Rightarrow\frac{d^2y}{dt^2}=-\frac{kdy}{dt}+k\left(-ky-\frac{dy}{dt}\right)-p^2y\quad\text{[From (i)]}$$

$$\Rightarrow\frac{d^2y}{dt^2}=-2k\frac{dy}{dt}-k^2y-p^2y$$

$$\Rightarrow\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+y(p^2+k^2)=0$$

SURE SHOT QUESTIONS

ARVIND ACADEMY

Chapter – 06 (Solution)

Application of Derivatives

➤ MCQ (1 mark)

1. Soln. (c): Let side of an equilateral triangle be x cm.

$\Rightarrow \quad \frac{dx}{dt} = 2\ cm/s$

$Area, \quad A = \frac{\sqrt{3}}{4}x^2$

Differentiating with respect to t, we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4}\, 2x\frac{dx}{dt}$$

Now, $\left[\frac{dA}{dt}\right]_{x=10} = \frac{\sqrt{3}}{4}\times 2\times 10\times 2 = 10\sqrt{3}\ cm^2/s$

2. Soln. (b): Given, $\frac{dy}{dt} = 10\,cm/s$, x = 2 m = 2000 cm and

Length of ladder = 5 m = 500 cm

$\Rightarrow \quad Let\ \angle ACB = \theta$

Now $\quad \sin\theta = \frac{y}{500}$

Differentiating w.r.t. t, we get

$$\cos\theta\frac{d\theta}{dt} = \frac{1}{500}\frac{dy}{dt}$$

$$\Rightarrow \quad \frac{200}{500}\times\frac{d\theta}{dt} = \frac{1}{500}\times 10$$

$$\Rightarrow \quad \frac{d\theta}{dt} = \frac{10}{200} \Rightarrow \frac{d\theta}{dt} = \frac{1}{20}\,radian/\text{sec}.$$

3. Soln. (b): Given $y = x^{1/5}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{5}x^{-4/5}\ and\ \left(\frac{dy}{dx}\right)_{(0,0)} = \frac{1}{5}(0) = 0$$

Hence, tangent is parallel to x-axis.

4. Soln. (c): If (α,β) is a point on the curve $3x^2 - y^2 = 8$ at which normal is parallel to given line, then $3\alpha^2 - \beta^2 = 8$ ....(i)

And $\left(\frac{dy}{dx}\right)_{(\alpha,\beta)} \times \left(-\frac{1}{3}\right) = -1 \Rightarrow \frac{3\alpha}{\beta}.\frac{1}{3} = 1 \Rightarrow \beta = \alpha$ ...(ii)

Solving (i) and (ii), we get $(\alpha,\beta) = (2,2)$ or $(-2,-2)$

Required normal are

$y - 2 = -\frac{1}{3}(x-2) \Rightarrow 3y + x - 8 = 0$

Or $y + 2 = -\frac{1}{3}(x+2) \Rightarrow 3y + x + 8 = 0$

5. Soln. (d): We have, $ay + x^2 = 7$

Differentiating w.r.t. x, we have

$$\Rightarrow \frac{ady}{dx} + 2x = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x}{a}$$

$$Now,\ \left(\frac{dy}{dx}\right)_{(1,1)} = \frac{-2}{a} \quad ....(i)$$

$$\text{Also, } x^3 = y \Rightarrow 3x^2 = \frac{dy}{dx}$$

$$\left(\frac{dy}{dx}\right)_{(1,1)} = 3 \quad ....(ii)$$

Since, curves cut orthogonally.

$$\therefore \quad \frac{-2}{a}\times 3 = -1 \Rightarrow a = 6$$

6. Soln. (a): Let $x = 2,\ x + \Delta x = 1.99$.

Then, $\Delta x = 1.99 - 2 = -0.01$

Let dx = $\Delta x = -0.01$

We have, $y = x^4 - 10$

$$\Rightarrow \frac{dy}{dx} = 4x^3 \Rightarrow \left(\frac{dy}{dx}\right)_{x=2} = 4(2)^3 = 32$$

$$\therefore \ \Delta y = \frac{dy}{dx}\Delta x = 32(-0.01) = -0.32$$

$$\Rightarrow \Delta y = -0.32 \ \ approximately$$

So, change in y = 0.32

7. Soln. (a): Given $y(1 + x^2) = 2 - x$ ....(i)

It crosses x-axis $\therefore$ $y = 0$

From (i), we have $0(1 + x^2) = 2 - x \Rightarrow x = 2$

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx}(1 + x^2) + y(2x) = -1$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1 - 2xy}{1 + x^2} \Rightarrow \left[\frac{dy}{dx}\right]_{(2,0)} = -\frac{1}{5}$$

$\therefore$ Equation of tangent at (2, 0) is

$$y - 0 = \frac{-1}{5}(x - 2) \Rightarrow x + 5y = 2$$

8. Soln. (d): Given

$$y = x^3 - 12x + 18 \Rightarrow \frac{dy}{dx} = 3x^2 - 12$$

Tangent is parallel to x-axis.

$$\Rightarrow \frac{dy}{dx} = 0 \Rightarrow 3x^2 - 12 = 0 \Rightarrow x = \pm 2$$

when $x = 2$ *when* $x = -2$

$y = 8 - 24 + 18 = 2$ $y = -8 + 24 + 18 = 34$

$\therefore$ Required points are (2, 2) and (-2, 34).

9. Soln. (b): Given, $y = e^{2x}$

$$\Rightarrow \frac{dy}{dx} = 2e^{2x} \Rightarrow \left[\frac{dy}{dx}\right]_{(0,1)} = 2$$

Equation of tangent at (0, 1) is

$$y - 1 = 2(x - 0) \Rightarrow y = 2x + 1$$

It meet x-axis $\therefore$ $y = 0 \Rightarrow x = \frac{-1}{2}$

Hence, required point is $\left(-\frac{1}{2}, 0\right)$.

10. Soln. (b): Given, $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$

At $(2, -1)$ $t^2 + 3t - 8 = 2$ ....(i)

And $2t^2 - 2t - 5 = -1$ ....(ii)

Solving (i) and (ii), we get t = 2

Now, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t - 2}{2t + 3}$

$$\therefore \left[\frac{dy}{dx}\right]_{t=2} = \frac{6}{7}$$

11. Soln. ([illegible]): Given, $x^3 - 3xy^2 + 2 = 0$

Differentiating w.r.t. x, we get

$$3x^2 - 3x(2y)\frac{dy}{dx} - 3y^2 = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2 - 3y^2}{6xy}$$

Also, $3x^2y - y^3 - 2 = 0$

Differentiating w.r.t. x, we get

$$3x^2\frac{dy}{dx} + 6xy - 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{6x}{3x^2 - 3y^2}\right)$$

Now, product of slopes =

$$\frac{3x^2 - 3y^2}{6xy} \times \left(\frac{-6xy}{3x^2 - 3y^2}\right) = -1$$

$\therefore$ They are perpendicular. Hence, angle $= \pi/2$

12. Soln. (b): We have, $f(x) = 2x^3 + 9x^2 + 12x - 1$

$$f'(x) = 6x^2 + 18x + 12 = 6(x^2 + 3x + 2) = 6(x+2)(x+1)$$

For decreasing $f'(x) \le 0$

$\therefore \quad 6(x+2)(x+1) \le 0$

$\Rightarrow \quad (x+2)(x+1) \le 0 \Rightarrow -2 \le x \le -1$

13. Soln. (d): We have, $f(x) = 2x + \cos x$

$f'(x) = 2 - \sin x$

We know that, $-1 \le \sin x \le 1$

$$-1 \le -\sin x \le 1 \Rightarrow 1 \le 2 - \sin x \le 3$$

$\Rightarrow f'(x) > 0 \;\therefore\;$ f(x) is always increasing.

14. Soln. (a): Given, $y = x(x-3)^2$

$$\Rightarrow \quad \frac{dy}{dx} = x.2(x-3) + (x-3)^2$$
$$= (x-3)(2x + x - 3) = (x-3)(3x-3) = 3(x-3)(x-1)$$

For decreasing $\frac{dy}{dx} < 0$

$\Rightarrow \quad (x-3)(x-1) < 0 \Rightarrow 1 < x < 3$

15. Soln. (c): $f(x) = \cos x \Rightarrow f'(x) = -\sin x$

In interval $\left(0, \frac{\pi}{2}\right)$, sin x is positive

$\therefore \quad f'(x) < 0 \;\forall\; x \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad f(x)$ is decreasing in $\left(0, \frac{\pi}{2}\right)$

16. Soln. (a): Since, f(x) = tanx – x

$\Rightarrow f'(x) = \sec^2 x - 1 = \tan^2 x$

$\tan^2 x$ *is always* $+ve$, *so* $f'(x) > 0$

$\therefore \quad$ f(x) always increases.

17. Soln. (c): Given,

$$f(x) = x^2 - 8x + 17 \Rightarrow f'(x) = 2x - 8$$

For minimum or maximum, we have f′(x) = 0

$\Rightarrow \; 0 = 2x - 8 \Rightarrow x = 4$

Now $f''(x) = 2$

At $x = 4$, $f''(x)$ is $+ve$

$\therefore \quad$ x = 4 is a point of local minima.

$\therefore \quad$ Minimum value

$= (4)^2 - 8(4) + 17 = 16 - 32 + 17 = 1$

18. Soln. (b): Given, $f(x) = x^3 - 18x^2 + 96x$

$\Rightarrow f'(x) = 3x^2 - 36x + 96$

$\therefore f'(x) = 0 \Rightarrow x^2 - 12x + 32 = 0 \Rightarrow x = 8, 4$

Now, $f(0) = 0$, $f(4) = 160$, $f(8) = 128$, $f(9) = 135$

So, smallest value of f(x) is 0 at x = 0.

19. Soln. (b): Given, $f(x) = \sin x.\cos x = \frac{\sin 2x}{2}$

$$f'(x) = \frac{1}{2}(\cos 2x).2 = \cos 2x$$

For maxima or minima, $f'(x) = 0$

$\Rightarrow \cos 2x = 0 \Rightarrow 2x = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$

Now, $f''(x) = -2\sin 2x$

At $x = \frac{\pi}{4}$, $f''(x) = -2\sin 2\left(\frac{\pi}{4}\right) = -2 < 0$

$\therefore \quad x = \frac{\pi}{4}$ is a point of local maxima.

$\therefore$ Max. value is

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4}.\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2}$$

20. Soln. (b): Since, $y = -x^3 + 3x^2 + 9x - 27$

$\therefore$ Slope, $m = \frac{dy}{dx} = -3x^2 + 6x + 9$

For maximum $\frac{dm}{dx} = -6x + 6$

Now, $\frac{dm}{dx}=0 \Rightarrow -6x+6=0 \Rightarrow x=1$

$\therefore \quad \frac{d^2m}{dx^2}=-6<0\ \forall\ x$

$\therefore \quad x=1$ is a point of local maximum.

$\therefore$ Maximum slope = $-3(1)^2+6(1)+9=12$

21. Ans. (b): We have, $f(x)=2x^3+9x^2+12x-1$

$\Rightarrow f'(x)=6x^2+18x+12$

For decreasing, f(x) < 0.

$\therefore \quad 6x^2+18x+12<0$

$\Rightarrow x^2+3x+2<0 \Rightarrow (x+1)(x+2)<0$

$\Rightarrow -2<x<-1$

So, f(x) is decreasing, if $x\in(-2,-1)$.

22. Ans. (c): $f(x)=x^3+3x$

For increasing, we must have f′(x) > 0

$\therefore f'(x)=3x^2+3>0$

$\Rightarrow (3x^2+1)>0$

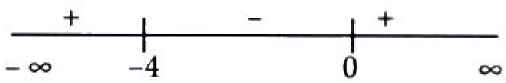
$\Rightarrow x^2+1>0$, *which is true* $\forall x\in R$.

23. Ans. (a): Given,

$y=x^3+6x^2+6 \Rightarrow \frac{dy}{dx}=3x^2+12x$

For increasing, $\frac{dy}{dx}>0 \Rightarrow 3x^2+12x>0$

$\Rightarrow 3x(x+4)>0$



So, y is strictly increasing in $(-\infty,-4)\cup(0,\infty)$.

24. Ans. (d): Let f(x) = x – sinx

Differentiating w.r.t. x, we get f′(x) = 1 – cosx

For function to be decreasing, f′(x) < 0

$\Rightarrow 1-\cos x<0 \Rightarrow \cos x>1$,

Which is not possible, because maximum value of cosx is 1.

$\therefore \quad f(x)=(x-\sin x)$ doesn't decrease at any value of x.

25. Ans. (b): Let f(x) = x – x²

$\therefore f'(x)=1-2x$

For critical point, $f'(x)=0$

$\Rightarrow 1-2x=0 \Rightarrow x=1/2$

Now, at x = ½, f″(x) = -2 < 0

So, f(x) has maximum value at x = ½.

26. Ans. (b): Let r be the radius, θ be the central angle and l be the length of the circular sector.

Given, $l+2r=20$

$\Rightarrow r\theta+2r=20\ (\because l=r\theta) \Rightarrow \theta=\frac{20-2r}{r}$

Let A be the area of the circular sector.

$\therefore A=\frac{\pi r^2\theta}{2\pi}=\frac{r^2}{2}.\left(\frac{20-2r}{r}\right)=r(10-r)$

$\Rightarrow \frac{dA}{dr}=10-2r$

For maximum or minimum value of A, we have

$\frac{dA}{dr}=0 \Rightarrow r=5$ *and* $\frac{d^2A}{dr^2}=-2<0$

$\therefore$ Area is maximum at r = 5

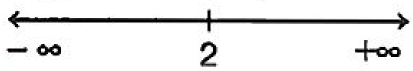
$\therefore$ Maximum area, A = 5(10 – 5) = 25 cm²

➤ Assertion-Reasoning (1 mark)

27. **Sol. (b)** Both A and R are true but R is not the correct explanation of A.

Explanation: We have, f(x) = x² - 4x + 6

or f′(x) = 2x - 4 = 2 (x - 2)



Therefore, f′(x) = 0 gives x = 2.

Now, the point x = 2 divides the real line into two disjoint intervals namely, (–∞, 2) and (2,∞).

In the interval (-∞, 2), f′(x) = 2x - 4 < 0.

Therefore, f is strictly decreasing in this interval.

Also, in the interval (2, ∞), f′(x) > 0 and so the function f is strictly increasing in this interval.

Hence, both the statements are true but Reason is not the correct explanation of Assertion.

28. Sol. **(b)** Both A and R are true but R is not the correct explanation of A.

Explanation: Assertion: We have,

For increasing function, $f'(x) \geq 0$

$\therefore 6(x^2 - 3x + 2) \geq 0$

$\Rightarrow 6(x - 2)(x - 1) \geq 0$

$\Rightarrow x \leq 1$ and $x \geq 2$

$\therefore$f(x) is increasing outside the interval (1, 2), therefore it is a true statement.

Reason: Now, $f'(x) < 0$

$\Rightarrow 6(x - 2)(x - 1) < 0$

$\Rightarrow 1 < x < 2$

$\therefore$ Assertion and Reason are both true but Reason is not the correct explanation of Assertion.

29. Sol. (a) Both A and R are true and R is the correct explanation of A.

Explanation: Both A and R are true and R is the correct explanation of A.

30. **Sol. (a)** Both A and R are true and R is the correct explanation of A.

Explanation: Let $f(x) = x^2 - 8x + 17$

$\therefore f'(x) = 2x - 8$

So, $f'(x) = 0$, gives $x = 4$

Here $x = 4$ is the critical number

Now, $f''(x) = 2 > 0, \forall x$

So, $x = 4$ is the point of local minima.

$\therefore$ Minimum value of f(x) at $x = 4$,

$f(4) = 4 \times 4 - 8 \times 4 + 17 = 1$

Hence, we can say that both Assertion and Reason are true and Reason is the correct explanation of the Assertion.

31. Sol. (b) Both A and R are true but R is not the correct explanation of A.

Explanation: We have,

$f(x) = 2x^3 - 6x^2 + 6x + 5$

$\Rightarrow f'(x) = 6x^2 - 12x + 6 = 6(x - 1)^2$

and $f''(x) = 12(x - 1)$

Now, $f'(x) = 0$ gives $x = 1$.

Also, $f''(1) = 0$.

Therefore, the second derivative test fails in this case.

So, we shall go back to the first derivative test.
Using first derivatives test, we get x = 1 is neither a point of local maxima nor a point of local minima and so it is a point of inflexion.

32. **Sol. (d)** A is false but R is true.

Explanation: Let $f(x) = 2x^3 - 24x$

$\Rightarrow f'(x) = 6x^2 - 24 = 6(x^2 - 4)$

$= 6(x + 2)(x - 2)$

For maxima or minima put $f'(x) = 0$.

$\Rightarrow 6(x + 2)(x - 2) = 0$

$\Rightarrow x = 2, -2$

We first consider the interval [1, 3].

So, we have to evaluate the value of f at the critical point x = 2 [1, 3] and at the end points of [1, 3].

At $x = 1$, $f(1) = 2 \times 1^3 - 24 \times 1 = -22$

At $x = 2$, $f(2) = 2 \times 2^3 - 24 \times 2 = -32$

At $x = 3$, $f(3) = 2 \times 3^3 - 24 \times 3 = -18$

$\therefore$The absolute maximum value of f(x) in the interval [1, 3] is -18 occurring at x = 3.

Hence, Assertion is false and Reason is true.

➢ CASE STUDY

33. 1. Ans. (c): $\because$ Volume of cylinder = $\pi r^2 h$

$\therefore$ V = volume of casted half cylinder = $(1/2)\pi r^2 h$

2. Ans. b): Total surface area, $S = \frac{2\pi r(r+h)}{2} + 2rh$

$= \pi r^2 + \pi rh + 2rh$

3. Ans. (c): Here,

$$S = \pi r^2 + \frac{2V(\pi+2)}{\pi r}\left[\because V = \frac{1}{2}\pi r^2 h \Rightarrow \frac{2V}{\pi r} = rh\right]$$

4. Ans. (a): $\because S = \pi r^2 + \frac{2V(\pi+2)}{\pi r}$

$$\Rightarrow \frac{dS}{dr} = 2\pi r - \frac{2V(\pi+2)}{\pi} \times \frac{1}{r^2}$$

For S to be minimum, $\frac{dS}{dr} = 0$

$$\Rightarrow 2\pi r = \frac{2V(\pi+2)}{\pi r^2} \Rightarrow \pi^2 r^3 = V(\pi+2)$$

5. Ans. (d): $\because V = \frac{1}{2}\pi r^2 h$ ....(i)

And S will be minimum, when $(\pi+2)V = \pi^2 r^3$

$\Rightarrow V = \frac{\pi^2 r^3}{\pi+2}$ ....(ii)

From (i) and (ii), we get

$$\Rightarrow \frac{1}{2}\pi r^2 h = \frac{\pi^2 r^3}{\pi+2} \Rightarrow \pi r^2 h(\pi+2) = 2\pi^2 r^3$$

$$\Rightarrow h(\pi+2) = 2\pi r \Rightarrow \frac{h}{2r} = \frac{\pi}{\pi+2}$$

Thus, required ratio i.e., h : 2r is $\pi : \pi + 2$.

34. 1. Ans. (d): Let F be the fuel cost per hour and v be the speed of train in km/hr.

According to question, we have,

$F \propto V^2 \Rightarrow F = kv^2$, where k is proportionality constant

$$\Rightarrow 48 = k(16)^2 \Rightarrow k = \frac{3}{16}$$

2. Ans. (b): Let total cost of running the train be C.

Then, $C = \frac{3}{16}v^2 t + 1200t$

Now, distance covered = 500 km $\Rightarrow Time = \frac{500}{v}$ hrs

$\therefore$ Total cost of running the train for 500 km

$$= \frac{3}{16}v^2\left(\frac{500}{v}\right) + 1200\left(\frac{500}{v}\right)$$

$$\Rightarrow C = \frac{375}{4}v + \frac{600000}{v}$$

3. Ans. (c): We have, $\frac{dC}{dv} = \frac{375}{4} - \frac{600000}{v^2}$

Put $\frac{dC}{dv} = 0 \Rightarrow v^2 = \frac{600000 \times 4}{375} = 6400$

$$\Rightarrow v = 80\,km/h$$

$$\frac{d^2C}{dv^2} = \frac{2 \times 600000}{v^3} > 0, for\ v = 80$$

$\therefore$ Most economical speed is 80 km/h.

4. Ans. (c): Fuel cost for running the train for 500 km

$$= \frac{3}{16}v^2\left(\frac{500}{v}\right)$$

$$= \frac{375}{4}v = \frac{375}{4} \times 80 = Rs.7500$$

5. Ans. (d) : Total cost for running the train for 500 km

$$= \frac{375}{4}v + \frac{600000}{v}$$

$$= \frac{375 \times 80}{4} + \frac{600000}{80} = Rs.15000$$

35. Ans. (i) (b): We have, perimeter of floor = 200 m

$$\Rightarrow 2x + 2\pi\left(\frac{Y}{2}\right) = 200$$

$$\Rightarrow 2x + \pi y = 200 \quad ......(i)$$

(ii)(a) : Area of rectangular region (A) = xy

$$= x\left(\frac{200-2x}{\pi}\right) \qquad [Using (i)]$$

$$= \frac{2}{\pi}(100x - x^2)$$

(iii)(c): We have, $A = \frac{2}{\pi}(100x - x^2)$

$$\Rightarrow \frac{dA}{dx} = \frac{2}{\pi}(100 - 2x)$$

For maximum or minimum, $\frac{dA}{dx} = 0$

$$\Rightarrow 100 - 2x = 0 \Rightarrow x = 50$$

Now, $\left[\frac{d^2A}{dx^2}\right]_{x=50} = -\frac{4}{\pi} < 0$

Thus, A is maximum at x = 50.

Thus, maximum value of $A = \frac{2}{\pi}[5000 - 2500]$

$$= \frac{5000}{\pi}m^2$$

(iv)(a) : Let P be the area of the whole floor.

Then,

$$P = xy + \pi\left(\frac{y}{2}\right)0^2 = xy + \frac{\pi}{4}y^2 = y\left(x + \frac{\pi}{4}y\right)$$

$$= \left(\frac{200-2x}{\pi}\right)\left(\frac{200+2x}{4}\right) \qquad \text{[Using (i)]}$$

$$= \frac{40000-4x^2}{4\pi} = \frac{10000-x^2}{\pi}$$

$$\therefore \frac{dP}{dx} = -\frac{2x}{\pi}$$

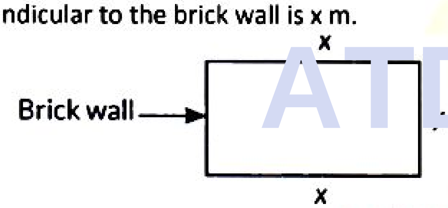
For maximum or minimum, $\frac{dP}{dx} = 0 \Rightarrow 0$

Now, $\frac{d^2P}{dx^2} = \frac{-2}{\pi} < 0$

So, P is maximum at x = 0m.

(v) (d)

36. Ans. Given, the length of side of garden perpendicular to the brick wall is x m.



The length of the side parallel to the brick wall is y m.

(i) 2x + y = 200

We know that area of rectangle is $= l \times b$

$$\Rightarrow A(x) = xy = x(200-2x) = 200x - 2x^2$$

(ii) Since, $A(x) = 200x - 2x^2$ .....(i)

Differentiating (i) w.r.t. x, we get

$$\frac{d}{dx}A(x) = 200 - 4x \qquad \text{..........(ii)}$$

For critical point $\frac{d}{dx}A(x) = 0$

$$\Rightarrow 200 - 4x = 0 \Rightarrow 4x = 200 \Rightarrow x = 50$$

Again differentiating (ii) w.r.t. x, we get

$\frac{d^2}{dx^2}A(x) = -4 < 0$ i.e., area A(x) is maximum at x = 50

Hence, maximum area is

$$A(50) = 200(50) - 2(50)^2$$
$$= 10000 - 5000 = 5000m^2$$

37. Ans. Let x be the side of square base and y be the height of the open tank.

$\therefore\ l = x,\ b = x\ and\ h = y$

Where $l,\ b\ and\ h$ be the length, breadth and height of tank respectively.

Volume of tank $V = x^2y \Rightarrow\ y = \frac{V}{x^2}$

The cost of the material will be least if the total surface area is least.

Total surface area of tank (S) = $x^2 + 4xy$

$$\Rightarrow S = x^2 + 4x\left(\frac{V}{x^2}\right) \quad \left(\because y = \frac{V}{x^2}\right)$$

$$\Rightarrow S = x^2 + \frac{4V}{x} \Rightarrow \frac{dS}{dx} = 2x - \frac{4V}{x^2}$$

For maxima or minima, $\frac{dS}{dx} = 0$

$$\Rightarrow 2x - \frac{4V}{x^2} = 0 \Rightarrow x^3 = 2V \Rightarrow x = 2y$$

$$(\because V = x^2y)$$

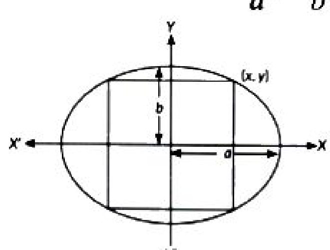
Also, $\frac{d^2S}{dx^2} = 2 + \frac{8V}{x^3} > 0$

$\therefore$ Cost of material is least, when $y = \frac{x}{2}$

i.e., the depth of the tank is half of its width.

As the cost is borne by nearby settled lower income families it shows that they are spending money on social welfare so that no body will face the water problem in future. It shows social responsibility.

38. Ans. (i) Given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Let $(x, y) = \left(x, \frac{b}{a}\sqrt{a^2 - x^2}\right)$ be the upper right vertex of the rectangle.

The area function $A = 2x \times 2\frac{b}{a}\sqrt{a^2 - x^2}$

$$= \frac{4b}{a}x\sqrt{a^2 - x^2},\ x \in (0, a)$$

(ii) The first derivative of function is

$$\frac{dA}{dx} = \frac{4b}{a}\left[x \times \frac{-x}{\sqrt{a^2 - x^2}} + \sqrt{a^2 - x^2}\right]$$

$$= \frac{4b}{a} \times \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}} = -\frac{4b}{a} \times \frac{2\left(x + \frac{a}{\sqrt{2}}\right)\left(x - \frac{a}{\sqrt{2}}\right)}{\sqrt{a^2 - x^2}}$$

To find the critical point, put $\frac{dA}{dx} = 0$

$$\Rightarrow x = \frac{a}{\sqrt{2}}$$

So, $x = \frac{a}{\sqrt{2}}$ is the critical point.

(iii) For the values of x less than $\frac{a}{\sqrt{2}}$ and close to $\frac{a}{\sqrt{2}}$, $\frac{dA}{dx} > 0$ and for the values of x greater than $\frac{a}{\sqrt{2}}$ and close to $\frac{a}{\sqrt{2}}$, $\frac{dA}{dx} < 0$.

Hence, by the first derivative test, there is a local maximum at the critical point $x = \frac{a}{\sqrt{2}}$.

Since there is only one critical point, therefore, the area of the soccer field is maximum at this critical point $x = \frac{a}{\sqrt{2}}$.

Thus, for maximum area of the soccer field, its length should be $a\sqrt{2}$ and its width should be $b\sqrt{2}$.

OR

$$A = 2x \times 2\frac{b}{a}\sqrt{a^2 - x^2},\ x \in (0, a)$$

Squaring both sides, we get

$$Z = A^2 = \frac{16b^2}{a^2}x^2(a^2 - x^2)$$

$$= \frac{16b^2}{a^2}(x^2a^2 - x^4),\ x \in (0, a)$$

A is maximum when Z is maximum

To find the critical point, put $\frac{dz}{dx} = 0$

$$\frac{dZ}{dx} = \frac{16b^2}{a^2}(2xa^2 - 4x^3)$$

$$= \frac{32b^2}{a^2} \times (a + \sqrt{2}x)(a - \sqrt{2}x)$$

To find the critical point, put $\frac{dZ}{dx} = 0 \Rightarrow x = \frac{a}{\sqrt{2}}$

The second derivative is:

$$\frac{d^2Z}{dx^2} = \frac{32b^2}{a^2}\left(a^2 - 6x^2\right)$$

$$\therefore \left(\frac{d^2Z}{dx^2}\right)_{x = \frac{a}{\sqrt{2}}} = \frac{32b^2}{a^2}\left(a^2 - 3a^2\right)$$

$$= -64b^2 < 0$$

Hence, by the second derivative test, there is a local maximum value of Z at the critical point $x = \frac{a}{\sqrt{2}}$.

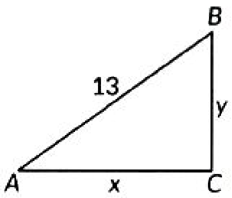
Thus, for maximum area of the soccer field, its length should be $a\sqrt{2}$ and its width should be $b\sqrt{2}$.

➤ Questions

39. Ans. Let foot of the ladder is at a distance x m from the wall and height on the wall is y m.

Here, $x^2+y^2=(13)^2$ [Using Pythagoras theorem]

Differentiating with respect to t, we get



$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

$$\Rightarrow \frac{dy}{dt}=\frac{-x}{y}\frac{dx}{dt}$$

When x = 5 m, $y^2=(13)^2-(5)^2=169-25=144$

$\therefore\ y=12m$

Also, $\frac{dx}{dt}=2\,cm/\sec$ [Given]

$$\therefore \frac{dy}{dt}=\frac{-5}{12}\times 2=\frac{-5}{6}cm/\sec$$

40. Ans. We have, $f(x)=\frac{x^4}{4}-x^3-5x^2+24x+12$ ....(i)

f(x) being polynomial function is continuous and derivable on R.

Differentiating (i) w.r.t. x, we get

$$f'(x)=\frac{4x^3}{4}-3x^2-10x+24$$

$$=x^3-3x^2-10x+24$$

$$=(x-2)(x^2-x-12)=(x-2)(x-4)(x+3)$$

(a) For strictly increasing, f′(x) > 0

$\Rightarrow (x-2)(x-4)(x+3)>0$

$\Rightarrow x\in(-3,2)\cup(4,\infty)$

(b) For strictly decreasing, f′(x) < 0

$\Rightarrow (x-2)(x-4)(x+3)<0$

$\Rightarrow x\in(-\infty,-3)\cup(2,4)$

41. Ans. We have, $f(x)=3x^4-4x^3-12x^2+5$

$$f'(x)=12x^3-12x^2-24x=12x(x^2-x-2)$$

$\Rightarrow f'(x)=12x(x+1)(x-2)$

Now, $f'(x)=0$

$\Rightarrow 12x(x+1)(x-2)=0$

$\Rightarrow x=-1, x=0$ or $x=2$

Hence, these points divide the whole real line into four disjoint open intervals namely $(-\infty,-1),(-1,0),(0,2)$ and $(2,\infty)$.

Interval	Sign of f′(x)	Nature of function
$(-\infty,-1)$	(-) (-) (-) < 0	Strictly decreasing
(-1, 0)	(-) (+) (-) > 0	Strictly increasing
(0, 2)	(+) (+) (-) < 0	Strictly decreasing
$(2,\infty)$	(+) (+) (+) > 0	Strictly increasing

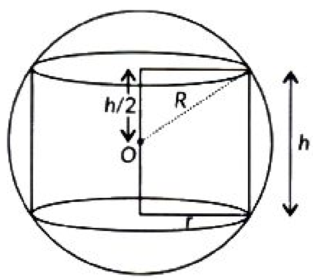
(a) F(x) is strictly increasing in $(-1,0)\cup(2,\infty)$.

F(x) is strictly decreasing in $(-\infty,-1)\cup(0,2)$.

42. Ans. Let r and h be the base radius and height of cylinder respectively.

$$\therefore \left(\frac{h}{2}\right)^2+r^2=R^2 \quad ....(i)$$

Now, V = Volume of the cylinder inscribed in a sphere = πr^2h



$$\Rightarrow V=\pi h\left(R^2-\frac{h^2}{4}\right) \quad \text{[Using (i)]}$$

$$\Rightarrow V=\pi\left(R^2h-\frac{h^3}{4}\right)$$

Now differentiating w.r.t. h, we get

$$\frac{dV}{dh}=\pi\left(R^2-\frac{3h^2}{4}\right) and \frac{d^2V}{dh^2}=\pi\left(0-\frac{3}{4}.2h\right)$$

For maximum or minimum,

$$\frac{dV}{dh}=0\Rightarrow R^2-\frac{3}{4}h^2=0$$

$$\Rightarrow h^2=\frac{4}{3}R^2\Rightarrow h=\frac{2R}{\sqrt{3}}$$

For this value of h,

$$\frac{d^2V}{dh^2}=-\frac{3}{2}\pi.\frac{2R}{\sqrt{3}}=-\sqrt{3}\pi R<0$$

$\Rightarrow$ V is maximum

Also maximum value of V

$$=\pi.\frac{2R}{\sqrt{3}}\left(R^2-\frac{1}{4}.\frac{4}{3}R^2\right)=\pi.\frac{2R}{\sqrt{3}}.\frac{2}{3}R^2$$

$$=\frac{4\pi}{3\sqrt{3}}R^3\, cu.units$$

43. Ans. Let ABC be a right angled triangle with BC = x, AC = y such that x + y = k, where k is any constant.

Let θ be the angle between the base and the hypotenuse.

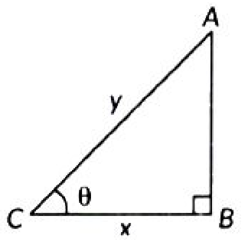
Let P be the area of the triangle.

$$P=\frac{1}{2}\times BC\times AB=\frac{1}{2}\times x\sqrt{y^2-x^2}$$

$$\Rightarrow P^2=\frac{x^2}{4}\left(y^2-x^2\right)$$

$$\Rightarrow P^2=\frac{x^2}{4}[(k-x)^2-x^2]$$

$$\Rightarrow P^2=\frac{k^2x^2-2kx^3}{4}$$



Let $Q=P^2$ i.e., $Q=\dfrac{k^2x^2-2kx^3}{4}$

$\therefore$ P is maximum when Q is maximum.

Differentiating Q w.r.t. x, we get

$$\frac{dQ}{dx}=\frac{2k^2x-6kx^2}{4} \qquad \text{........(i)}$$

For maximum or minimum area,

$$\frac{dQ}{dx}=0\Rightarrow k^2x-3kx^2=0\Rightarrow x=\frac{k}{3}$$

Differentiating (i) w.r.t. x, we get

$$\frac{d^2Q}{dx^2}=\frac{2k^2-12kx}{4}$$

$$\therefore \left[\frac{d^2Q}{dx^2}\right]_{x=\frac{k}{3}}=\frac{-k^2}{2}<0$$

Thus, Q is maximum when $x=\dfrac{k}{3}$

$\Rightarrow$ P is maximum at $x=\dfrac{k}{3}$

Now, $x=\dfrac{k}{3}\Rightarrow y=k-\dfrac{k}{3}=\dfrac{2k}{3}$ $[\because\ x+y=k]$

$$\therefore \cos\theta=\frac{x}{y}=\frac{k/3}{2k/3}=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{3}$$

So, the area of ΔABC is maximum when angle between the hypotenuse and base is $\dfrac{\pi}{3}$.

44. *Soln.* We have, f(x) = -|x – 1| + 5 for all $x\in R$

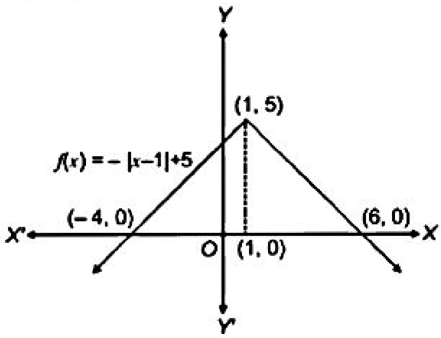
Clearly, $|x-1|\geq 0$ for all $x\in R$

Taking minus sign both side, we get

$-|x-1|\leq 0$ for all $x\in R$

$-|x-1|+5\leq 5$ for all $x\in R$

$\Rightarrow f(x)\leq 5$ for all $x\in R$



So, 5 is the maximum value of f(x).

Now, f(x) = 5

$\Rightarrow\quad -|x-1|+5=5$

$\Rightarrow\quad |x-1|=0\Rightarrow x=1$

Thus, f(x) attains the maximum value 5 at x = 1.
Since f(x) can be made as small as possible.
Therefore the minimum value of f(x) does not exist.

45. Soln. Given function is

$$f(x)=\frac{x^4}{4}-x^3-5x^2+24x+12$$

$$\Rightarrow f'(x)=\frac{4x^3}{4}-3x^2-10x+24$$

For critical points, put f′(x) = 0

$$\therefore \quad x^3-3x^2-10x+24=0$$

$$(x-2)(x^2-x-12)=0$$

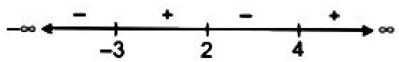
$$(x-2)(x-4)(x+3)=0$$

$$\Rightarrow \quad x=2,4,-3$$

Therefore, we have the intervals

$(-\infty,-3),(-3,2),(2,4)$ and $(4,\infty)$

Since f′(x) > 0 in $(-3,2)\cup(4,\infty)$

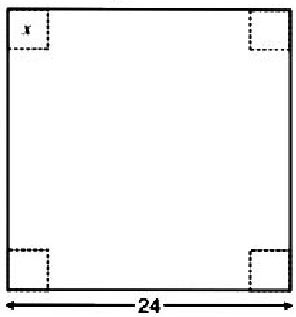


$\therefore f(x)$ is increasing in interval $(-3,2)\cup(4,\infty)$

And $f'(x)<0\, in\, (-\infty,-3)\cup(2,4)$

$\therefore\ f(x)$ is decreasing in $(-\infty,-3)\cup(2,4)$.

46. Sol. Let length of a side of the square cut out = x cm



Volume, $\quad V=(24-2x)^2x$

$$\frac{dV}{dx}=2(24-2x)(-2)x+(24-2x)^2$$

$$=(24-2x)(-4x+24-2x)$$

$$=(24-2x)(24-6x)$$

$$=12(12-x)(4-x)$$

Put $\frac{dV}{dx}=0$ i.e., $(12-x)(4-x)=0$

Since, $\quad x\neq 12,$

$\therefore \quad x=4$

Hence, the volume will be maximum when length of a side of the square cut out = 4 cm.

47. Soln. Let the length, breadth and height of the open tank be x, x and y units respectively.

Then, $\quad$ Volume (V) = x^2y

..........(i)

Total surface area (S) = $x^2 + 4xy$ ....(ii)

$$S=x^2+4x\frac{V}{x^2} \qquad \text{[using (i)]}$$

$$\Rightarrow \quad S=x^2+\frac{4V}{x}$$

$$\Rightarrow \quad \frac{dS}{dx}=2x-\frac{4V}{x^2}$$

For critical points, put

$$\frac{dS}{dx}=0$$

$$\Rightarrow \quad 2x-\frac{4V}{x^2}=0$$

$$\Rightarrow \quad 2x^3=4V$$

$$\Rightarrow \quad 2x^3=4x^2y \qquad \text{[using(i)]}$$

$$\Rightarrow \quad x=2y \qquad ....(iii)$$

$$\text{Now,} \quad \frac{d^2S}{dx^2}=2+\frac{8V}{x^3}$$

$$=2+\frac{8V}{8y^3} \qquad \text{[using (iii)]}$$

$$=2+\frac{V}{y^3}>0$$

Area is minimum, thus cost is minimum when x = 2y.

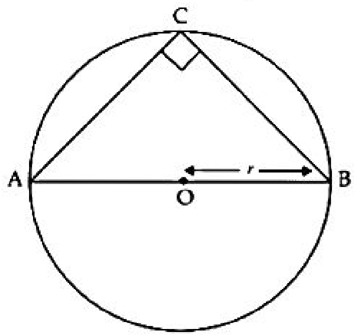
i.e., depth of tank is half of the width.

Value: Any relevant value.

48. Soln.

Let r be the radius of the circuit then,

AB = 2r (AB is diameter)



Let, BC = x units

We know that angle subtended by diameter in a circle is right angle

$$\therefore \quad \angle C=90^0$$

Then, $\quad AC=\sqrt{(AB)^2-(BC)^2}$

$$AC=\sqrt{(2r)^2-(x)^2}=\sqrt{4r^2-x^2}$$

.........(i)

Now, area of ΔABC

$$A=\frac{1}{2}(AC)(BC)$$

$$A=\frac{1}{2}\sqrt{4r^2-x^2}\,(x)$$

Differentiating A w.r.t. x, we get

$$\Rightarrow \quad \frac{dA}{dx}=\frac{1}{2}\left\{\sqrt{4r^2-x^2}+x\frac{1}{2\sqrt{4r^2-x^2}}.\frac{d}{dx}(4r^2-x^2)\right\}$$

$$\Rightarrow \quad \frac{dA}{dx}=\frac{1}{2}\left\{\sqrt{4r^2-x^2}-\frac{x^2}{\sqrt{4r^2-x^2}}\right\}$$

$$=\frac{1}{2}\left\{\frac{4r^2-x^2-x^2}{\sqrt{4r^2-x^2}}\right\}$$

$$\Rightarrow \quad \frac{dA}{dx}=\frac{1}{2}\left\{\frac{4r^2-2x^2}{\sqrt{4r^2-x^2}}\right\}$$

The critical numbers of x are given by $\frac{dA}{dx}=0$

$$\Rightarrow \quad \frac{1}{2}\left\{\frac{4r^2-2x^2}{\sqrt{4r^2-x^2}}\right\}=0$$

$$\Rightarrow \quad 4r^2-2x^2=0$$

$$\Rightarrow \quad 4r^2=2x^2$$

$$\therefore \quad x=\sqrt{2}r$$

$$Now, \quad \frac{dA}{dx}=\frac{1}{2}\left\{\frac{4r^2-2x^2}{\sqrt{4r^2-x^2}}\right\}$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2A}{dx^2}=\frac{1}{2}\left\{\begin{array}{l}(-4x)\dfrac{1}{\sqrt{4r^2-x^2}}+(4r^2-2x^2)\\ \qquad\left(\dfrac{-1}{2}\right)(4r^2-x^2)^{-3/2}\dfrac{d}{dx}(4r^2-x^2)\end{array}\right\}$$

$$\Rightarrow \quad \frac{d^2A}{dx^2}=\frac{1}{2}\left\{\frac{-4x}{\sqrt{4r^2-x^2}}+\frac{x(4r^2-2x^2)}{(4r^2-x^2)^{3/2}}\right\}$$

$$\Rightarrow \quad \left(\frac{d^2A}{dx^2}\right)_{x=\sqrt{2}r}=\frac{1}{2}\left\{\frac{-4(\sqrt{2}r)}{\sqrt{4r^2-2r^2}}+\frac{\sqrt{2}r(4r^2-4r^2)}{(4r^2-2r^2)^{3/2}}\right\}$$

$$=\frac{-2\sqrt{2}r}{\sqrt{2}r}=-2<0$$

Thus, A is maximum when x = $\sqrt{2}r$.

Putting $x=\sqrt{2}r$ in (i),

$$AC=\sqrt{4r^2-(\sqrt{2}r)^2}$$

$$\therefore \quad AC=\sqrt{2}r$$

$$\therefore \quad BC=AC=\sqrt{2}\,r$$

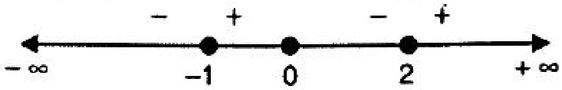
Hence, A is maximum when the triangle is isosceles.

49. Soln. $\quad f'(x)=12x^3-12x^2-24x$

$$=12x(x+1)(x-2)$$

$$f'(x)>0,\ \forall x\in(-1,0)\cup(2,\infty)$$

$$f'(x)<0,\ \forall x\in(-\infty,-1)\cup(0,2)$$



$\therefore$ $f(x)$ is strictly increasing in $(-1,0)\cup(2,\infty)$

And strictly decreasing in $(-\infty,-1)\cup(0,2)$

50. Soln. We know that, a function y = f(x) is said to be increasing on R, if $\frac{dy}{dx}>0,\forall x\in R$

Given, $\quad y=x^3-3x^2+3x$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=3x^3-6x+3$$

Or $\quad \frac{dy}{dx}=3(x^2-2x+1)$

Or $\quad \frac{dy}{dx}=3(x-1)^2$

Now, $3(x-1)^2>0$ for all real values of x, i.e., $\forall\ x\in R$

$$\frac{dy}{dx}>0,\ \forall\ x\in R$$

Hence, the given function is increasing or R.

51. Soln. Let l,b,h be the length, breadth and depth of the tank, respectively.

$$\therefore \quad l\times b\times 3=75$$

$$or \quad l\times b=25$$

Let C be the cost, then

$$C=100(l\times b)+50\times 2[h(b+l)]$$

$$=100\left(l\times\frac{25}{l}\right)+300\left(\frac{25}{l}+l\right)$$

$$=2500+300\left(\frac{25}{l}+l\right)$$

Differentiating w.r.t. l,

$\therefore \quad \frac{dC}{dl} = 0 + 300\left(\frac{-25}{l^2} + 1\right)$

Putting $\quad \frac{dC}{dl} = 0,$

or $\quad 300\left(-\frac{25}{l^2} + 1\right) = 0$

or $\quad l^2 = 25 \, or \, l = 5$

Getting $\quad \frac{d^2C}{dl^2} = 300\left(\frac{50}{l^3}\right)$

Or $\quad \left(\frac{d^2C}{dl^2}\right)_{at\, l=5} = \frac{15000}{125} > 0$

i.e., C is minimum when $l = 5$

$\Rightarrow \quad b = 5$

$\therefore \quad C = 100(25) + 300(10)$

$= 2,500 + 3,000$

$= 5,500$

Hence the minimum cost is Rs.5,500.

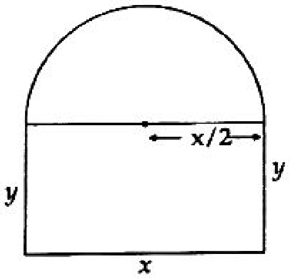
52. Soln. Let length and breadth of rectangle be x and y.

$\therefore \quad P = 2y + x + \pi\frac{x}{2}$ (Given)

$\therefore \quad A = xy + \frac{1}{2}\pi\frac{x^2}{4}$

$= \frac{x}{2}\left[P - x - \frac{\pi x}{2}\right] + \frac{\pi x^2}{8}$

$= P\frac{x}{2} - \frac{x^2}{2} - \pi\frac{x^2}{4} + \pi\frac{x^2}{8}$



$\frac{dA}{dx} = \frac{P}{2} - x - \frac{\pi x}{2} + \frac{\pi x}{4}$

$= \frac{P}{2} - x - \frac{\pi x}{4}$

$\frac{dA}{dx} = 0$

Or $\quad x = \frac{2P}{4+\pi} \, and \, y = \frac{P}{4+\pi}$

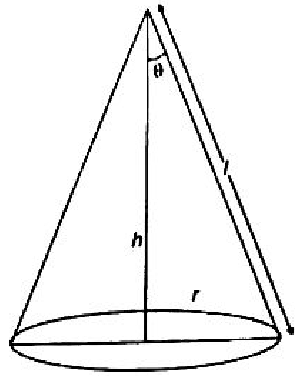
$\frac{d^2A}{dx2} = -1 - \pi/4 < 0$

On Area is maximum, when length

$= \frac{2P}{4+\pi}$'

Breadth $= \frac{P}{4+\pi}$

53. Soln.



$r = l\sin\theta$

$h = l\cos\theta$

$V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}l^3\sin^2\theta\cos\theta$

Or $\quad \frac{dV}{d\theta} = \frac{\pi}{3}l^3[2\sin\theta\cos^2\theta - \sin^3\theta]$

For maxima and minima, $\frac{dV}{d\theta} = 0$

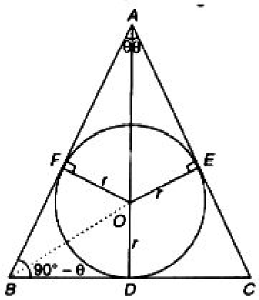
$\sin\theta(2\cos^2\theta - \sin^2\theta) = 0$

Or $\quad \cos\theta = \frac{1}{\sqrt{3}} \, or \, \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\because \quad \frac{d^2V}{d\theta^2}$ is negative.

Hence, volume of the cone is maximum when semi-vertical angle is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

54. Soln. Let ΔABC be an isosceles triangle with AB = AC.



Let $\angle BAC = 2\theta$
AO bisects $\angle BAC$.
Join OE, OF, OD,
Where O is the centre of the circle and OE = OF = OD = r
Also, $OE \perp AC$, $OD \perp BC$ and $OF \perp AB$.

In ΔAOE, $\tan\theta = \dfrac{r}{AE}$

Or $AE = r\cot\theta$

Similarly $AF = r\cot\theta$

In $\angle ABD$, $AD \perp BC$ (ΔABC is isosceles)

$$\therefore \quad \angle ABD = 90^0 - \theta = \frac{\pi}{2} - \theta$$

OB bisects $\angle ABD$, $\therefore \angle OBF = \angle OBD = \dfrac{\pi}{4} - \dfrac{\theta}{2}$

In ΔOFB,

Or $BF = r\cot\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right) = r\cot\left(\dfrac{\pi - 2\theta}{4}\right)$

Similarly $BD = DC = CE = r\cot\left(\dfrac{\pi - 2\theta}{4}\right)$

We have, perimeter of ΔABC

$$P = AB + BC + CA$$
$$= AE + EC + BD + DC + AF + BF$$
$$= 2r\cot\theta + 4r\cot\left(\frac{\pi - 2\theta}{4}\right)$$

Differentiate with respect to θ,

$$\therefore \quad \frac{dP}{d\theta} = 2r(-\mathrm{cosec}^2\theta)$$
$$+4r\left(-\mathrm{cosec}^2\left(\frac{\pi - 2\theta}{4}\right) \times -\frac{1}{2}\right)$$
$$= -2r\,\mathrm{cosec}^2\theta + 2r\,\mathrm{cosec}^2\left(\frac{\pi - 2\theta}{4}\right)$$

Or On equating, $\dfrac{dP}{d\theta} = 0 \Rightarrow 2r\,\mathrm{cosec}^2\left(\dfrac{\pi - 2\theta}{4}\right)$

$$= 2r\,\mathrm{cosec}^2 gq$$

Or $\sin^2\left(\dfrac{\pi - 2\theta}{4}\right) = \sin^2\theta$

Or

$$\sin\left(\frac{\pi - 2\theta}{4}\right) = \sin\theta \text{ or } \sin\left(\frac{\pi - 2\theta}{4}\right) = -\sin\theta$$

But $0 \le \theta \le \dfrac{\pi}{4}$ or $\sin 2\theta \le \sin\theta$

$$\therefore \quad \sin\left(\frac{\pi - 2\theta}{4}\right) \ne -\sin\theta$$

$$\& \quad \sin\left(\frac{\pi - 2\theta}{4}\right) = \sin\theta$$

or $\dfrac{\pi - 2\theta}{4} = \theta$

or $\pi - 2\theta = 4\theta$ or $\pi = 6\theta$

or $\theta = \dfrac{\pi}{6}$ or $2\theta = \dfrac{\pi}{3}$

Or ΔABC is an equilateral triangle.
By second derivative test,

$$\frac{d^2P}{d\theta^2} = -2r\{2\,\mathrm{cosec}\,\theta(-\mathrm{cosec}\,\theta\cot\theta)\} + 2r$$
$$\left[2\,\mathrm{cosec}\left(\frac{\pi - 2\theta}{4}\right)\left\{-\mathrm{cosec}\left(\frac{\pi - 2\theta}{4}\right)\cot\left(\frac{\pi - 2\theta}{4}\right)\right\}\right] \times \left(-\frac{1}{2}\right)$$
$$= 4r\,\mathrm{cosec}^2\theta\cot\theta + \frac{4r}{2}\mathrm{cosec}^2\left(\frac{\pi - 2\theta}{4}\right)\cot\left(\frac{\pi - 2\theta}{4}\right)$$

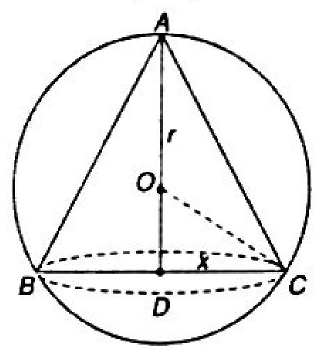
$$\therefore \quad \frac{d^2P}{d\theta} \ge 0 \text{ at } \theta = \frac{\pi}{3}$$

$\therefore$ Perimeter is minimum when $\theta = \dfrac{\pi}{3}$

$$P = 2r\cot\frac{\pi}{6} + 4r\cot\left(\frac{\pi - 2(\pi/6)}{4}\right)$$
$$= r\sqrt{3} + 4r\cot\left(\frac{2\pi}{12}\right)$$
$$= 2r\sqrt{3} + 4r\sqrt{3}$$
$$\therefore \quad P = 6\sqrt{3}r$$

55. Soln. Let radius of cone be x and its height be h.

$\therefore \quad OD = (h - r)$



Volume of cone (V)

$$= \frac{1}{3}\pi x^2 h \quad \text{...........(i)}$$

In $\Delta OCD, x^2 + (h-r)^2 = r^2$ or $x^2 = r^2 - (h-r)^2$

$\therefore \quad V = \frac{1}{3}\pi h\{r^2 - (h-r)^2\}$

$= \frac{1}{3}\pi(-h^3 + 2h^2 r)$

Or $\quad \frac{dV}{dh} = \frac{\pi}{3}(-3h^2 + 4hr)$

$\therefore \quad \frac{dV}{dh} = 0 \; or \; h = \frac{4r}{3}$

$\frac{d^2V}{dh^2} = \frac{\pi}{3}(-6h + 4r)$

$= \frac{\pi}{3}\left(-6\left(\frac{4r}{3}\right) + 4r\right)$

$= -\frac{4\pi r}{3} < 0$

$\therefore \quad at \; h = \frac{4r}{3}$, volume is maximum

Maximum volume

$= \frac{1}{3}\pi\left\{-\left(\frac{4r}{3}\right)^3 + 2\left(\frac{4r}{3}\right)^2 r\right\}$

$= \quad \frac{8}{27}\cdot\left(\frac{4}{3}\pi r^3\right)$

$= \quad \frac{8}{27}$ (Volume of sphere)

56. Soln. Let the radius and height of cylinder be r and h respectively

$\therefore \quad V = \pi r^2 h \quad$ ....(i)

But $\; r^2 = R^2 - \frac{h^2}{4}$

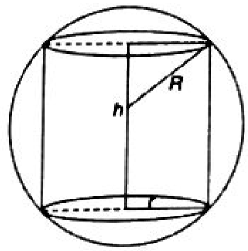
$\therefore \quad \pi h\left(R^2 - \frac{h^2}{4}\right) = \pi\left(R^2 h - \frac{h^3}{4}\right)$

or $\quad \frac{dV}{dh} = \pi\left(R^2 - \frac{3h^2}{4}\right)$

For maximum or minimum

$\therefore \quad \frac{dV}{dh} = 0 \; or \; h^2 = \frac{4R^2}{3}$

Or $\quad h = \frac{2R}{\sqrt{3}}$



And $\; \frac{d^2V}{dh^2} = \pi\left(-\frac{6h}{4}\right) < 0$

Maximum volume = $\pi.\left[R^2.\frac{2R}{\sqrt{3}} - \frac{1}{4}\left(\frac{2R}{\sqrt{3}}\right)^3\right]$

$= \frac{4\pi R^3}{3\sqrt{3}}$ cubic units

57. Soln. Let ABC be right-circular cone having radius 'r' and height 'h'. If V and S are its volume and surface area (curved) respectively, then

$S = \pi r l$

Or $\quad S = \pi r\sqrt{h^2 + r^2} \quad$ ....(i)

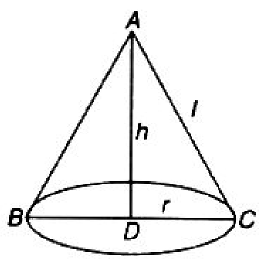
Also, $V = \frac{1}{3}\pi r^2 h \; or \; h = \frac{3V}{\pi r^2}$

Putting the value of h in (i), we get

$S = \pi r\sqrt{\frac{9V^2}{\pi^2 r^4} + r^2}$

Or $\quad S^2 = \pi^2 r^2\left(\frac{9V^2 + \pi^2 r^6}{\pi^2 r^4}\right)$

[Maxima or Minima is same for S or S^2]



Or $\quad S^2 = 9\frac{V^2}{r^2} + \pi^2 r^4$

Differentiating with respect to 'r'

$(S^2)' = \frac{-18V^2}{r^3} + 4\pi^2 r^3$

..........(ii)

Now, $\quad (S^2)' = 0$

Or $\quad \frac{-18V^2}{r^3} + 4\pi^2 r^3 = 0$

Or $4\pi^2 r^6 = 18V^2$

Putting value of V,

$$4\pi^2 r^6 = 18 \times \frac{1}{9}\pi^2 r^4 h^2$$

Or $2r^2 = h^2$

Or $r = \dfrac{h}{\sqrt{2}}$

Differentiating (ii) with respect to 'r', again

$$\left(S^2\right)'' = \frac{54V^2}{r^4} + 12\pi^2 r^2$$

Or $\left.\left(S^2\right)''\right]_{r=\frac{h}{\sqrt{2}}} > 0$ (for any value of r)

Hence, S^2 i.e., S is minimum for

$$r = \frac{h}{\sqrt{2}}$$

Or $h = \sqrt{2}r$.

i.e., for least curved surface, altitude is equal to $\sqrt{2}$ times the radius of the base.

Then,

If θ is the semi – vertical angle of cone, then

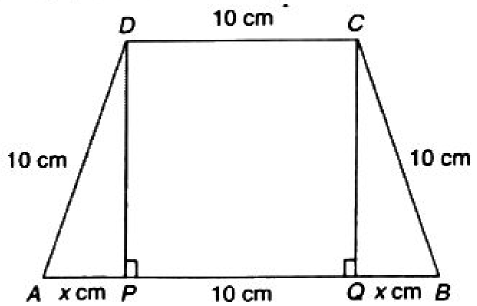
$$\cot\theta = \frac{h}{r} = \frac{\sqrt{2}r}{r}$$

Or $\cot\theta = \sqrt{2}$

Or $\theta = \cot^{-1}\left(\sqrt{2}\right)$

58. Soln. The required trapezium is as given in figure. Draw perpendiculars DP and CQ on AB. Let AP = x cm. Note that $\Delta APD \cong \Delta BQC$. Therefore, QB = x cm. Also, by Pythagoras theorem DP = QC = $\sqrt{100 - x^2}$.

Let A be the area of the trapezium.



Then, A $\equiv$ A(x)

$$= \frac{1}{2}(sum\ of\ parallel\ sides) \times (\text{height})$$

$$= (x+10)\sqrt{100 - x^2}$$

Or

$$A'(x) = (x+10)\frac{(-2x)}{2\sqrt{100-x^2}} + \left(\sqrt{100-x^2}\right)$$

$$= \frac{-2x^2 - 10x + 100}{\sqrt{100 - x^2}}$$

Now, A'(x) = 0 gives $2x^2 + 10x - 100 = 0$,

i.e., $x = 5$ *and* $x = -10$

So, $x = 5$.

Now,

$$A''(x) = \frac{\sqrt{100-x^2}(-4x-10) - (2x^2 - 10x + 100)\dfrac{(-2x)}{2\sqrt{100-x^2}}}{100 - x^2}$$

$$= \frac{2x^3 - 300x - 1000}{\left(100 - x^2\right)^{\frac{1}{2}}}$$ (on simplification)

Or $A''(5) = \dfrac{2(5)^3 - 300(5) - 1{,}000}{\left(100 - (5)^2\right)^{\frac{3}{2}}}$

$$= \frac{-2{,}250}{75\sqrt{75}} = \frac{-30}{\sqrt{75}} < 0$$

Thus, area of trapezium is maximum at x = 5 and the maximum area is given by

$$A(5) = (5+10)\sqrt{100 - (5)^2}$$

$$= 15\sqrt{75} = 75\sqrt{3}\ \text{cm}^2$$

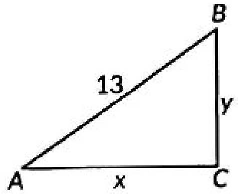
59. Soln. Let foot of the ladder is at a distance x from the wall and height on the wall is y.

Here, $x^2 + y^2 = (13)^2$ [Using Pythagoras theorem]

Differentiating with respect to t, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}$$



When $x = 5m,\ y^2 = (13)^2 - (5)^2 = 169 - 25 = 144$

$\therefore \quad y = 12m$

Also, $\frac{dx}{dt} = 2\,cm/\sec$ [Given]

$\therefore \quad \frac{dy}{dt} = \frac{-5}{12} \times 2 = \frac{-5}{6}$ cm/sec

60. Soln. Let 'a' be the side of an equilateral triangle.

Then $\frac{da}{dt} = 2\,cm/\sec$

Let 'A' be the area of an equilateral triangle, then

$$A = \frac{\sqrt{3}}{4}a^2 \Rightarrow \frac{dA}{dt} = 2 \times \frac{\sqrt{3}}{4}a\frac{da}{dt} = \frac{\sqrt{3}}{2}a\frac{da}{dt}$$

$$\therefore \left(\frac{dA}{dt}\right)_{a=20} = \frac{\sqrt{3}}{2} \times 20 \times 2 = 20\sqrt{3}\,cm^2/\sec$$

61. Soln. Let 'a' be the side of an equilateral triangle.

Then $\frac{da}{dt} = 2\,cm/\sec$

Let 'A' be the area of an equilateral triangle, then

$$A = \frac{\sqrt{3}}{4}a^2 \Rightarrow \frac{dA}{dt} = 2 \times \frac{\sqrt{3}}{4}a\frac{da}{dt} = \frac{\sqrt{3}}{2}a\frac{da}{dt}$$

$$\therefore \left(\frac{dA}{dt}\right)_{a=10} = \frac{\sqrt{3}}{2} \times 10 \times 2 = 10\sqrt{3}\,cm^2/\sec$$

62. Soln. Let r, S and V respectively be the radius, surface area and volume of sphere at any time t.

Then, $\frac{dV}{dt} = 3\,cm^3/\sec$

Now, $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{3}{4\pi r^2}$

Also, $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

$\Rightarrow \frac{dS}{dt} = \frac{6}{r}$

$$\therefore \left(\frac{dS}{dt}\right)_{r=2} = \frac{6}{2} = 3\,cm^2/\sec$$

63. Soln. We have,

$C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$

$\Rightarrow \frac{dC}{dx} = 0.015x^2 - 0.04 + 30$

Now,

$$\left(\frac{dC}{dx}\right)_{x=3} = 0.015 \times 3^2 - 0.04 \times 3 + 30 = 30.015$$

64. Soln. We have, $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ ....(i)

f(x) being polynomial function is continuous and derivable on R.

differentiating (i) w.r.t. x, we get

$$f'(x) = \frac{4x^3}{4} - 3x^2 - 10x + 24$$

$$= x^3 - 3x^2 - 10x + 24$$

$$= (x-2)(x^2 - x - 12) = (x-2)(x-4)(x+3)$$

(a) For increasing, $f'(x) > 0$

$\Rightarrow \quad (x-2)(x-4)(x+3) > 0$

$\Rightarrow \quad x \in (4, \infty) \cup (-3, 2)$

(b) For decreasing, $f'(x) < 0$

$\Rightarrow (x-2)(x-4)(x+3) < 0$

$\Rightarrow \ x \in (2, 4) \cup (-\infty, -3)$

65. Soln.

$f(x) = \sin 3x - \cos 3x \Rightarrow f'(x) = 3\cos 3x + 3\sin 3x$

$f'(x) = 0 \Rightarrow 3\cos 3x = -3\sin 3x$

$\Rightarrow \ \cos 3x = -\sin 3x \Rightarrow \tan 3x = -1$

Which gives $3x = \frac{3\pi}{4}\ or\ \frac{7\pi}{4}\ or\ \frac{11\pi}{4}$

$\Rightarrow \ x = \frac{\pi}{4}\ or\ \frac{7\pi}{12}\ or\ \frac{11\pi}{12}$ $[\because\ 0 < x < \pi]$

The points $x = \frac{\pi}{4}$, $x = \frac{7\pi}{12}$ and $x = \frac{11\pi}{12}$ divide the interval $(0, \pi)$ into four disjoint intervals,

$$\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{7\pi}{12}\right), \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right), \left(\frac{11\pi}{12}, \pi\right)$$

Now, $f'(x) > 0$ in $\left(0, \frac{\pi}{4}\right)$

$\Rightarrow$ f is strictly increasing in $\left(0, \frac{\pi}{4}\right)$

$f'(x) < 0\ in\ \left(\frac{\pi}{4}, \frac{7\pi}{12}\right)$

$\Rightarrow$ f is strictly decreasing in $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right)$

$f'(x) > 0$ in $\left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$

$\Rightarrow$ f is strictly increasing in $\left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$

$f'(x) < 0$ in $\left(\frac{11\pi}{12}, \pi\right)$

$\Rightarrow$ f is strictly decreasing in $\left(\frac{11\pi}{12}, \pi\right)$

Hence, f is strictly increasing in the intervals

$\left(0, \frac{\pi}{4}\right) \cup \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$

And f is strictly decreasing in the intervals

$\left(\frac{\pi}{4}, \frac{7\pi}{12}\right) \cup \left(\frac{11\pi}{12}, \pi\right)$

66. Soln. Here, $f(x) = x^2 - x + 1;\ x \in (-1, 1)$

$\Rightarrow f'(x) = 2x - 1$

$f'(x) = 0 \Rightarrow x = \frac{1}{2}$

Now $f'(x) = 2\left(x - \frac{1}{2}\right) > 0\ for\ \frac{1}{2} < x < 1$

$\Rightarrow$ f is strictly increasing in $\left(\frac{1}{2}, 1\right)$

Also $f'(x) = 2\left(x - \frac{1}{2}\right) < 0$ for $-1 < x < \frac{1}{2}$

$\Rightarrow$ f is strictly decreasing in $\left(-1, \frac{1}{2}\right)$

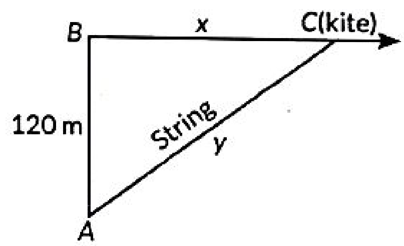
Thus f is neither increasing nor decreasing in (-1, 1).

67. Soln. We have, $y^2 = x^2 + (120)^2$

$$\Rightarrow \quad 2y\frac{dy}{dt} = 2 \times \frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dy}{dt} = 52\frac{x}{y}$$



Putting $y = 130\ in\ y^2 = x^2 + (120)^2$, we get x = 50.

$$\therefore \quad \frac{dy}{dt} = \frac{52 \times 50}{130} = 20$$

68. Soln. Given x and y are the sides of two squares such that $y = x - x^2$.

Let area of the first square $(A_1) = x^2$

And area of the second square

$(A_2) = y^2 = (x - x^2)^2$

Now, $\frac{dA_1}{dt} = \frac{d}{dt}x^2 = 2x.\frac{dx}{dt}$

Also, $\frac{dA_2}{dt} = \frac{d}{dt}(x - x^2)^2$

$$= 2(x - x^2)\left(\frac{dx}{dt} - 2x.\frac{dx}{dt}\right) = \frac{dx}{dt}(1 - 2x)2(x - x^2)$$

$$\therefore \frac{dA_2}{dA_1} = \frac{(dA_2/dt)}{(dA_1/dt)} = \frac{\frac{dx}{dt}.(1 - 2x)(2x - 2x^2)}{2x.\frac{dx}{dt}}$$

$$= \frac{(1 - 2x)2x(1 - x)}{2x} = (1 - 2x)(1 - x)$$

$$= 2x^2 - 3x + 1$$

SURE SHOT QUESTIONS

Chapter – 07 (Solution)

Integrals

➤ MCQ (1 mark)

1. Soln. We have, $\int \frac{\cos 2x-\cos 2\theta}{\cos x-\cos\theta}dx$

$$=\int \frac{2\sin(x+\theta).\sin(\theta-x)}{2\sin\left(\frac{x+\theta}{2}\right).\sin\left(\frac{\theta-x}{2}\right)}dx$$

$$=\int \frac{2\sin\left(\frac{x+\theta}{2}\right).\cos\left(\frac{x+\theta}{2}\right).2\sin\left(\frac{\theta-x}{2}\right).\cos\left(\frac{\theta-x}{2}\right)}{\sin\left(\frac{x+\theta}{2}\right).\sin\left(\frac{\theta-x}{2}\right)}dx$$

$$=4\int \cos\left(\frac{x+\theta}{2}\right).\cos\left(\frac{\theta-x}{2}\right)dx$$

$$=2\int(\cos\theta+\cos x)\,dx=2(x\cos\theta+\sin x)+C$$

2. Soln. (c): We have, $\int \frac{dx}{\sin(x-a)\sin(x-b)}$

$$=\frac{1}{\sin(b-a)}\int \frac{\sin(b-a)}{\sin(x-a)\sin(x-b)}dx$$

$$=\frac{1}{\sin(b-a)}\int \frac{\sin((x-a)-(x-b))}{\sin(x-a)\sin(x-b)}dx$$

$$=\frac{1}{\sin(b-a)}\int \frac{\sin(x-a).\cos(x-b)-\cos(x-a)\sin(x-b)}{\sin(x-a)\sin(x-b)}dx$$

$$=\frac{1}{\sin(b-a)}\int(\cot(x-b)-\cot(x-a))dx$$

$$=\frac{1}{\sin(b-a)}[\log|\sin(x-b)-\log|\sin(x-a)|]+C$$

$$=\operatorname{cosec}(b-a)\left[\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right|\right]+C$$

3. Soln. (a): We have $\int 1.\tan^{-1}\sqrt{x}dx$

$$=\tan^{-1}\sqrt{x}.(x)-\int\frac{1}{1+x}\times\frac{1}{2\sqrt{x}}\times x\,dx$$

$$=x\tan^{-1}\sqrt{x}-\int\frac{1}{1+x}\times\frac{1}{2\sqrt{x}}\times x\,dx$$

$$=x\tan^{-1}\sqrt{x}-\int\frac{x}{(1+x)2\sqrt{x}}dx$$

$$=x\tan^{-1}\sqrt{x}-\int\left(\frac{1+x}{(1+x)2\sqrt{x}}-\frac{1}{(1+x)2\sqrt{x}}\right)dx$$

$$=x\tan^{-1}\sqrt{x}-\int\frac{dx}{2\sqrt{x}}+\int\frac{dx}{2\sqrt{x}(1+x)}$$

$$=x\tan^{-1}\sqrt{x}-\sqrt{x}+\tan^{-1}\sqrt{x}+C$$

$$=(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}+C$$

4. Soln. (a): Let $I=\int e^x\left(\frac{1-x}{1+x^2}\right)^2dx$

$$=\int e^x\left(\frac{1+x^2-2x}{(1+x^2)^2}\right)dx=\int e^x\left(\frac{1}{1+x^2}-\frac{2x}{(1+x^2)^2}\right)dx$$

Above integral is of the type $\int e^x f(x)+f'(x))dx$

$\therefore$ Solution is $e^x f(x)+C \quad \therefore \quad I=e^x\left(\frac{1}{1+x^2}\right)+C$

5. Soln. (d): Let $I=\int\frac{x^9}{(4x^2+1)^6}dx$

$$=\int\frac{x^9dx}{x^{12}\left(4+\frac{1}{x^2}\right)^6}=\int\frac{dx}{x^3\left(4+\frac{1}{x^2}\right)^6}$$

Putting $4+\frac{1}{x^2}=t\Rightarrow\frac{-2}{x^3}dx=dt$, we get

$$I = -\frac{1}{2}\int \frac{dt}{t^6} = \frac{-1}{2}\int t^{-6}dt$$

$$= -\frac{1}{2}\left(\frac{t^{-5}}{-5}\right) + C = \frac{1}{10}\left(4 + \frac{1}{x^2}\right)^{-5} + C$$

6. Soln. (c): We have, $\int \frac{dx}{(x+2)(x^2+1)}$

Let $\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$

$\Rightarrow 1 = A(x^2+1) + Bx(x+2) + C(x+2)$ .....(i)

Putting x = 0 in (i), we get A + 2C = 1

Putting x = -2 in (i), we get $A = \frac{1}{5} \Rightarrow C = \frac{2}{5}$

Putting x = 1 in (i), we get

$1 = 2A + 3B + 3C$, we get $B = \frac{-1}{5}$

$$\therefore \int \frac{dx}{(x+2)(x^2+1)} = \frac{1}{5}\int \frac{dx}{x+2} - \frac{1}{5}\int \frac{x\,dx}{x^2+1} + \frac{2}{5}\int \frac{dx}{x^2+1}$$

$$= \frac{1}{5}\log|x+2| - \frac{1}{10}\log|x^2+1| + \frac{2}{5}\tan^{-1}x + C$$

Hence, $a = \frac{-1}{10}$ *and* $b = \frac{2}{5}$

7. Soln. (d): We have, $\int \frac{x^3}{x+1}dx = \int \frac{x^3-1+1}{x+1}dx$

$$= \int\left(\frac{(x+1)(x+1-x)}{x+1} - \frac{1}{x+1}\right)dx$$

$$= \int (x^2+1-x)dx - \int \frac{dx}{x+1} = \frac{x^3}{3} + x - \frac{x^2}{2} - \log|x+1| + C$$

8. Soln. (d): We have, $\int \frac{x+\sin x}{1+\cos x}dx$

$$= \int \frac{x}{1+\cos x}dx + \int \frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}dx$$

$$= \int \frac{x}{2\cos^2(x/2)}dx + \int \tan\frac{x}{2}dx$$

$$= \int \frac{1}{2}\sec^2\frac{x}{2}.x\,dx + \int \tan\frac{x}{2}dx$$

$$= \frac{1}{2}\left[x.\tan\frac{x}{2}.2 - \int \tan\frac{x}{2}.2dx\right] + \int \tan\frac{x}{2}dx + C$$

$$= x.\tan\frac{x}{2} - \int \tan\frac{x}{2}dx + \int \tan\frac{x}{2}dx + C = x.\tan\frac{x}{2} + C$$

9. Soln. (d): Let $I = \int \frac{x^3dx}{\sqrt{1+x^2}}$

Putting $x^2 = t \Rightarrow 2x\,dx = dt$, we get

$$I = \frac{1}{2}\int \frac{t\,dt}{\sqrt{1+t}} = \frac{1}{2}\int \frac{t+1-1}{\sqrt{1+t}}dt$$

$$= \frac{1}{2}\int\left(\sqrt{1+t} - (1+t)^{-1/2}\right)dt$$

$$= \frac{1}{2}\times\frac{2}{3}(1+t)^{3/2} - \frac{1}{2}\times 2(1+t)^{1/2} + C$$

$$= \frac{1}{3}(1+x^2)^{3/2} - (1+x^2)^{1/2} + C$$

$Hence, a = \frac{1}{3}$, **b** $= -1$

10. Soln. (a): Let $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{1+\cos 2x}$

Here, $f(x) = 1 + \cos 2x$ and

$$f(-x) = 1 + \cos(-2x) = 1 + \cos 2x = f(x)$$

$$\Rightarrow \quad f(x) = f(-x)$$

Hence, f(x) is even function.

$$\therefore\ I = 2\int_0^{\pi/4} \frac{dx}{1+\cos 2x} = 2\int_0^{\pi/4} \frac{dx}{2\cos^2 x} = 2\int_0^{\pi/4} \frac{1}{2}\sec^2 x\,dx$$

$$= |\tan x|_0^{\pi/4} = \left(\tan\frac{\pi}{4} - \tan 0\right) = 1$$

11. Soln. (d): We have, $\int_0^{\pi/2} \sqrt{1-\sin 2x}\,dx$

$$= \int_0^{\pi/2} \sqrt{\cos^2 x + \sin^2 x - 2\sin x \cos x}\,dx = \int_0^{\pi/2} |\cos x - \sin x|\,dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x)dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)dx$$

$[\because 0 < x < \pi/4, \cos x > \sin x \Rightarrow \cos x - \sin x > 0$ and $\pi/4 < x < \pi/2, \sin x > \cos x \Rightarrow \sin x - \cos x > 0]$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= [(\sin \pi/4 + \cos \pi/4) - (\sin 0 + \cos 0) - [(\cos \pi/2 + \sin \pi/2) - (\cos \pi/4 + \sin \pi/4)]$$

$$= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right] - \left[1 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right]$$

$$= \frac{2}{\sqrt{2}} - 1 - 1 + \frac{2}{\sqrt{2}} = \frac{4}{\sqrt{2}} - 2 = 2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$$

12. Soln. (b): Let $I = \int \frac{\sec x}{\sec x - \tan x} dx$

$$= \int \frac{\sec x(\sec x + \tan x)}{(\sec x - \tan x)(\sec x + \tan x)} dx$$

$$= \int \left(\frac{\sec^2 x + \sec x \tan x}{\sec^2 x - \tan^2 x} \right) dx$$

$$= \int \sec^2 x dx + \int \sec x \tan x\, dx \quad [\because \sec^2 x - \tan^2 x = 1]$$

$$= \tan x + \sec x + c$$

13. Soln.

(b): Let $I = \int e^{5\log x} dx$

$$= \int e^{\log x^5} dx = \int x^5 dx \quad [\because e^{\log x} = x]$$

$$= \frac{x^6}{6} + C$$

14. Soln.

(a): Let $I = \int x^2 e^{x^3} dx$

Put $x^3 = t \Rightarrow 3x^2 dx = dt$

$$\therefore I = \int e^t \frac{dt}{3} = \frac{1}{3} e^t + C = \frac{1}{3} e^{x^3} + C$$

10. Soln.

(a): Let $I = \int \frac{e^x(1+x)}{\cos^2(xe^x)} dx$

Put $xe^x = t$

$\Rightarrow (xe^x + e^x)dx = dt$

$\Rightarrow e^x(x+1)dx = dt$

$\therefore I = \int \frac{dt}{\cos^2 t} = \int \sec^2 t dt = \tan t + c = \tan(xe^x) + c$

11. Soln.

(d) Let $I = \int e^x \left(\log x + \frac{1}{x} \right) dx$

$\Rightarrow I = e^x \log x + c$

$\left(\because \int e^x [f(x) + f'(x)] dx = e^x f(x) + c \right)$

12. Soln.

(d): Let $\int \frac{e^x}{x+1} [1 + (x+1)\log(x+1)] dx$

$$= \int e^x \left[\frac{1}{x+1} + \log(x+1) \right] dx$$

It is of the form $\int e^x [f(x) + f'(x) dx]$

Where f(x) = $\log(x+1)$ and $f'(x) = \frac{1}{x+1}$

So, $I = e^x \log(x+1) + C$

13. Soln.

(d): Let $I = \int_{-1}^{1} \frac{|x-2|}{x-2} dx$

$$= \int_{-1}^{1} \frac{-(x-2)}{x-2} dx = \int_{-1}^{1} -1.dx = [-x]_{-1}^{1}$$

$$= -[1 - (-1)] = -2$$

14. Soln.

(a): Let $I=\int_0^4 (e^{2x}+x)dx=\left[\frac{e^{2x}}{2}+\frac{x^2}{2}\right]_0^4$

$$=\frac{e^8}{2}+\frac{16}{2}-\frac{e^0}{2}-0$$

$$=\frac{e^8}{2}+\frac{16}{2}-\frac{1}{2}=\frac{e^8+15}{2}$$

15. Soln.

(a): Let $I=\int_0^{\pi/8}\tan^2(2x)dx=\int_0^{\pi/8}(\sec^2(2x)-1)dx$

$$=\left(\frac{1}{2}\tan 2x-x\right)_0^{\pi/8}=\frac{1}{2}\tan 2\left(\frac{\pi}{8}\right)-\frac{\pi}{8}$$

$$=\frac{1}{2}\tan\frac{\pi}{4}-\frac{\pi}{8}$$

$$=\frac{1}{2}-\frac{\pi}{8}=\frac{4-\pi}{8}$$

16. Soln.

(d): Let $I=\int_{-\pi/4}^{\pi/4}\sec^2 x dx=[\tan x]_{-\pi/4}^{\pi/4}$

$$=\tan\frac{\pi}{4}-\tan\left(-\frac{\pi}{4}\right)=1+1=2$$

➢ Assertion-Reasoning (1 mark)

17. Soln.

(a): Let $I=\int_2^8\frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx$ ....(i)

$$=\int_2^8\frac{\sqrt{10-(10-x)}}{\sqrt{10-x}+\sqrt{10-(10-x)}}dx$$

$$\left(\because\int_a^b f(x)dx=\frac{b}{a}f(a+b-x)dx\right)$$

$$=\int_2^8\frac{\sqrt{x}}{\sqrt{10-x}+\sqrt{x}}dx \quad ....(ii)$$

Adding (i) and (ii), we get

$$2I=\int_2^8\frac{\sqrt{10-x}+\sqrt{x}}{\sqrt{x}+\sqrt{10-x}}dx=\int_2^8 1dx=[x]_2^8$$

$$\Rightarrow I=\frac{1}{2}(8-2)=\frac{6}{2}=3$$

Hence, both assertion and reason are true and reason is the correct explanation of assertion.

18. Sol. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation: Let y = x^x

$\Rightarrow$ log y = x log x

Differentiating w.r.t. x

$\Rightarrow \frac{1}{y}\frac{dy}{dx}=x\left(\frac{1}{x}\right)+\log x(1)$

$\frac{dy}{dx}=y(1+\log x)$

$=x^x(1+\log x)$

Hence R is true.

Since $\frac{d}{dx}(x^x)=x^x(1+\log x)$

$\int x^x(1+\log x)\,dx=x^x+c$

Using the concept of anti-derivative, A is true.

R is the correct explanation for A.

19. Sol. **(d)** A is false but R is true.

Explanation: $\int\frac{dx}{1+x^2}$

Let x = $\tan\theta$

dx = $\sec^2\theta\, d\theta$

$=\int\frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)}$

$=\int\frac{(1+\tan^2\theta)d\theta}{(1+\tan^2\theta)}$ $\{\because \sec^2\theta=1+\tan^2\theta\}$

$=\theta$

$=\tan^{-1}x+c$

20. Sol.

(c) A is true but R is false.

Explanation: $\int\sec^4 x\tan x\ dx$

$=\int\sec^2 x\cdot\sec^2 x+\tan x\ dx$

$=\int\left(1+\tan^2 x\right)\sec^2 x\tan x\ dx$ $\{\because 1+\tan^2 x=\sec^2 x\}$

$=\int(1+t)t\sec^2 x\,dx$ {Let t = tan x $\frac{dt}{dt}=\sec^2 x$, dt = $\sec^2 x$ dx}

$=\int\left(1+t^2\right)t\,dt$

$=\int t\,dt+\int t^3 dt$

$=\frac{t^2}{2}+\frac{t^2}{a}+c$

$=\frac{\tan^2 x}{2}+\frac{\tan^4 x}{4}+c$

➤ Case Study Question

21. Sol.

(i) Let $I = \int \frac{2x+5}{x^2+5x-7}dx$

let $x^2 + 5x - 7 = t$

$\Rightarrow d(x^2 + 5x - 7) = dt$

$\Rightarrow 2x + 5 = dt$

$\therefore I = \int \frac{dt}{t} = \log|t| + c$

$= \log|x^2 + 5x - 7)| + c$

(ii) Let $I = \int \frac{1}{x(3+\log x)}dx$

let $3 + \log x = t$

$\Rightarrow d(3 + \log x) = dt$

$\Rightarrow \frac{1}{x}dx = dt$

$\therefore I = \int \frac{dt}{t} = \log|t| + c$

$= \log|(3 + \log x)| + c$

(iii) Let $I = \int \frac{1}{1+e^{-x}}dx$

or $I = \int \frac{e^x}{e^x+1}dx$ (on dividing N^r and D^r by e^{-x})

let $e^x + 1 = t$

$\Rightarrow d(e^x + 1) = dt$

$\Rightarrow e^x dx = dt$

$\therefore I = \int \frac{dt}{t} = \log|t| + c$

$= \log|1 + e^x| + c$

(iv) Let $I = \int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx$

let $e^x + e^{-x} = t$

$\Rightarrow d(e^x + e^{-x}) = dt$

$\Rightarrow (e^x - e^{-x})\,dx = dt$

$\therefore I = \int \frac{dt}{t} = \log|t| + c$

$= \log|(e^x + e^{-x})| + c$

22. Sol.

(i) $\int \frac{\sin(\tan^{-1}x)}{1+x^2}dx$

Let $\tan^{-1} x = t$

Differentiating both sides w.r.t.x

$\frac{1}{1+x^2} = \frac{dt}{dx}$ (Using $\frac{d(\tan^{-1}x)}{dx} = \frac{1}{1+x^2}$)

$\frac{dx}{1+x^2} = dt$

Now,

Integrating the function

$\int \frac{\sin(\tan^{-1}x)}{1+x^2}dx$

Putting $\tan^{-1} x = t$ & $\frac{dx}{1+x^2}$ dt

$= \int \sin t \,.\, dt$

$= -\cos t + C$

putting back $t = \tan^{-1} x$

$= -\cos(\tan^{-1} x) + C$

(ii) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx$

Let $\cos x = t$

Differentiating both sides w.r.t.x

$-\sin x = \frac{dt}{dx}$

$\sin x\,dx = -dt$

Now,

$\int \frac{\sin x}{\cos x}dx$

Putting $\cos x = t$ & $\sin x\,dx = dt$

$= \int \frac{1}{t}\,.\,(-dt)$

$= -\int \frac{dt}{t}$

$= -\log|t| + C$

Putting back $t = \cos x$

$= -\log|\cos x| + C$

$= \log|\cos x|^{-1} + C$

$= \log \frac{1}{|\cos x|} + C$

$= \log|\sec x| + C$

(iii) $\int \frac{2x}{1+x^2}dx$

Let $1 + x^2 = t$

Differentiating both sides w.r.t.x

$2x = \frac{dt}{dx}$

$2x\,dx = dt$

Now,

Integrating the function

$\int \frac{2x}{1+x^2}dx$

Putting $1 + x^2 = t$ & $2xdx = dt$

$= \int \frac{1}{t}\,.\,dt$

$= \log|t| + C$

Putting back $t = 1 + x^2$

$= \log|1+ x^2| + C$

(iv) $\int \frac{1}{x+x\log x}dx = \int \frac{1}{x(1+\log x)}dx$

Let 1 + log x = t

Differentiate w.r.t.x

$0 + \frac{dt}{dx} = \frac{1}{x}$

$0 + \frac{dt}{dx} = \frac{1}{x}$

dt.x = dx

Now,

$\int \frac{1}{x+x\log x} \cdot dx = \int \frac{1}{x(1+\log x)} \cdot dx$

Putting 1 + log x & dx = x dt

$= \int \frac{1}{x(t)}$ dt . x

$= \int \frac{1}{t}$ dt

= log|t| + C

Putting back t = 1 + log x

= log|1 + log x| + C

➢ Questions

1. Soln.

Let $I = \int \frac{\sin x}{\sin(x-2a)}dx$

$Put \quad x-2a=t$

$\Rightarrow \quad x = 2a+t \Rightarrow dx = dt$

$\therefore \quad I = \int \frac{\sin(t+2a)}{\sin t}dx$

$= \int \frac{(\sin t\cos 2a + \cos t \sin 2a)}{\sin t}dx$

$= \int (\cos 2a + \cot t . \sin 2a)dx$

$= t\cos 2a + \sin 2a \log|\sin t| + c$

$= (x-2a)\cos 2a + \sin 2a \log|\sin(x-2a)| + c$

2. Soln.

Let $I = \int \frac{(3\sin\theta - 2)\cos\theta}{5-\cos^2\theta - 4\sin\theta}d\theta$

(Put $\cos^2\theta = 1-\sin^2\theta$)

$= 3\int \frac{\sin\theta\cos\theta}{4+\sin^2\theta-4\sin\theta}d\theta - 2\int \frac{\cos\theta}{4+\sin^2\theta - 4\sin\theta}d\theta$

$= 3I_1 - 2I_2$ (say) ....(i)

Now, $I_1 = \int \frac{\sin\theta\cos\theta}{4+\sin^2\theta-4\sin\theta}d\theta$

Put $\sin^2\theta = t \Rightarrow 2\sin\theta\cos\theta d\theta = dt$

$\therefore \quad I_1 = \frac{1}{2}\int \frac{dt}{4+t-4\sqrt{t}} = \frac{1}{2}\int \frac{dt}{(\sqrt{t}-2)^2}$

Put $\sqrt{t} - 2 = u \Rightarrow \sqrt{t} = u+2$

$\Rightarrow \frac{1}{2\sqrt{t}}dt = du \Rightarrow dt = 2(u+2)du$

$\therefore \quad I_1 = \int \frac{(u+2)}{u^2}du = \int \frac{du}{u} + 2\int \frac{du}{u^2}$

$= \log u - \frac{2}{u} + C_1 = \log(\sqrt{t}-2) - \frac{2}{\sqrt{t}-2} + C_1$

$= \log(\sin\theta - 2) - \frac{2}{\sin\theta - 2} + C_1$ .....(ii)

Also, $I_2 = \int \frac{\cos\theta}{4+\sin^2\theta - 4\sin\theta}d\theta$

Put $\sin\theta = m \Rightarrow \cos\theta d\theta = dm$

$\therefore I_2 = \int \frac{dm}{4+m^2-4m} = \int \frac{dm}{(m-2)^2}$ ......(iii)

From (i), (ii) and (iii), we get

$I = 3\log(\sin\theta - 2) - \frac{6}{\sin\theta-2} + \frac{2}{\sin\theta - 2} + C$,

Where C = $3C_1 - 2C_2$

$\Rightarrow I = 3\log(\sin\theta - 2) - \frac{4}{\sin\theta-2} + C$

3. Soln.

Let $I = \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}dx = \int_{\pi/6}^{\pi/3} \frac{\sin x+\cos x}{\sqrt{1-(1-\sin 2x)}}dx$

$= \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}}dx$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)dx = dt$

And when, $x = \frac{\pi}{6}, then\ t = \frac{1-\sqrt{3}}{2} = -\alpha$

$\therefore I = \int_{-\alpha}^{\alpha} \frac{dt}{\sqrt{1-t^2}}$

$= [\sin^{-1}t]_{-\alpha}^{\alpha} = 2\sin^{-1}\alpha = 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$

4. Soln.

Let $$I=\int_0^{\pi} e^{2x}.\sin\left(\frac{\pi}{4}+x\right)dx$$

Put $\frac{\pi}{4}+x=t \Rightarrow x=t-\frac{\pi}{4} \Rightarrow dx=dt$

When $x=0, t=\frac{\pi}{4}$ *and When* $x=\pi, t=\frac{5\pi}{4}$

$$\therefore I=\int_{\pi/4}^{5\pi/4} e^{2\left(t-\frac{\pi}{4}\right)}\sin t\,dt = e^{-\pi/1}\int_{\pi/4}^{5\pi/4} e^{2t}\sin t dt$$

$$=e^{-\pi/2}\left[\left(\sin t\frac{e^{2t}}{2}\right)_{\pi/4}^{5\pi/4}-\int_{\pi/4}^{5\pi/4}\cos\frac{e^{2t}}{2}dt\right]$$

$$=e^{-\pi/2}\left[\frac{1}{2}\left(e^{5\pi/2}\sin\frac{5\pi}{4}-e^{\pi/2}\sin\frac{\pi}{4}\right) -\left(\frac{e^{2t}}{4}\cos t\right)_{\pi/4}^{5\pi/4}-\int_{\pi/4}^{5\pi/4}\frac{e^{2t}}{4}\sin t dt\right]$$

$$=e^{-\pi/2}\left[\frac{1}{2}\left(\frac{-1}{\sqrt{2}}e^{5\pi/2}-\frac{1}{\sqrt{2}}e^{\pi/2}\right) -\frac{1}{4}\left(-\frac{1}{\sqrt{2}}e^{5\pi/2}-\frac{1}{\sqrt{2}}e^{\pi/2}\right)-\frac{I}{4}\right]$$

$$\Rightarrow I+\frac{1}{4}I=-\frac{1}{2\sqrt{2}}[e^{2\pi}+1]+\frac{1}{4\sqrt{2}}[e^{2\pi}+1]$$

$$\Rightarrow \frac{5}{4}I=\frac{(e^{2\pi}+1)}{2\sqrt{2}}\left[\frac{1}{2}-1\right]$$

$$=-\frac{1}{4\sqrt{2}}[e^{2\pi}+1] \Rightarrow I=\frac{-1}{5\sqrt{2}}(1+e^{2\pi})$$

5. Soln.

Let

$$I=\int_0^{\pi/4}\frac{dx}{\cos^3 x\sqrt{2\sin 2x}}=\int_0^{\pi/4}\frac{dx}{\cos^3 x\sqrt{2.2\sin x\cos x}}$$

$$=\frac{1}{2}\int\frac{dx}{\cos^{\left(3+\frac{1}{2}\right)}x.\sin^{\frac{1}{2}}x}$$

$$=\frac{1}{2}\int_0^{\pi/4}\frac{dx}{\cos^{\frac{7}{2}}x.\tan^{\frac{1}{2}}x.\cos^{\frac{1}{2}}x}$$

$$=\frac{1}{2}\int_0^{\pi/4}\frac{dx}{\cos^4 x\sqrt{\tan x}}=\frac{1}{2}\int_0^{\pi/4}\frac{\sec^4 x}{\sqrt{\tan x}}dx$$

Put $\tan x=t \Rightarrow \sec^2 x dx=dt$

When x = 0, then t = 0 and when $x=\frac{\pi}{4}$ then t = 1

$$\therefore I=\frac{1}{2}\int_0^1\frac{(1+t^2)\,dt}{\sqrt{t}}=\frac{1}{2}\int_0^1(t^{-\frac{1}{2}}+t^{\frac{3}{2}})dt$$

$$=\frac{1}{2}\left[\frac{t^{1/2}}{1/2}+\frac{t^{5/2}}{5/2}\right]_0^1=\frac{1}{2}\left[2+\frac{2}{5}\right]$$

$$=1+\frac{1}{5}=\frac{6}{5}$$

6. Soln. We have,

$$\int\frac{dx}{\sqrt{5-4x-2x^2}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\frac{5}{2}-2x-x^2}}$$

$$=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\frac{5}{2}-(2x+x^2)}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\frac{5}{2}+1-(1+2x+x^2)}}$$

[Add and subtract 1]

$$=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\frac{7}{2}-(x+1)^2}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\left(\sqrt{\frac{7}{2}}\right)^2-(x+1)^2}}$$

$$=\frac{1}{\sqrt{2}}\sin^{-1}\frac{x+1}{\sqrt{\frac{7}{2}}}+c\quad\left[\because\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}+c\right]$$

$$=\frac{1}{\sqrt{2}}\sin^{-1}\left[\frac{\sqrt{2}(x+1)}{\sqrt{7}}\right]+c$$

7. Soln. Let $I=\int\frac{1-x^2}{x(1-2x)}dx$

$$=\int\frac{1-x^2}{x-2x^2}dx$$

$\because$ deg (num) = deg (den)

$\therefore$

$$\begin{array}{r|l} & \;\; 1/2 \\ -2x+x^2 & -x^2+1 \\ & -x^2+\frac{1}{2}x \\ & \;+\quad - \\ \hline & -\frac{1}{2}x+1 \end{array}$$

$$I=\int\left(\frac{1}{2}+\frac{-\frac{1}{2}x+1}{x(1-2x)}\right)dx$$

$$I = \frac{1}{2}x + \int \frac{-\frac{1}{2}x+1}{x(1-2x)}dx \qquad ....(i)$$

Let $I_1 = \dfrac{-\frac{1}{2}x+1}{x(1-2x)} = \dfrac{A}{x} + \dfrac{B}{1-2x}$

$$-\frac{1}{2}x + 1 = A(1-2x) + Bx$$

...........(ii)

If x = 0

Then from equation (ii)

1 = A

If $x = \frac{1}{2}$

Then from equation (ii)

$$-\frac{1}{4} + 1 = \frac{B}{2}$$

$$\frac{-1+4}{4} = \frac{B}{2}$$

$$\frac{3}{4} = \frac{B}{2}$$

$\therefore \quad B = \frac{3}{2}$

$\therefore \quad I_1 = \int \frac{1}{x}dx + \frac{3}{2}\int \frac{dx}{1-2x}$

$$= \log|x| + \frac{3}{2(-2)}\int \frac{-2}{1-2x}dx$$

$$= \log|x| - \frac{3}{4}\log|1-2x|$$

$$\left[\because \int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c\right]$$

$\therefore \quad I = \frac{1}{2}x + \log|x| - \frac{3}{4}\log|1-2x| + c$

8. Soln. $I = \int \dfrac{2\sin x \cos x}{(1+\sin x)(2+\sin x)}dx$

Put $\sin x = t$

Differentiate both sides w.r.t. x

$$\cos x dx = dt$$

$\therefore \quad I = \int \dfrac{2t}{(1+t)(2+t)}dt$

Now, $\dfrac{2t}{(1+t)(2+t)} = \dfrac{A}{1+t} + \dfrac{B}{2+t}$

$$2t = A(2+t) + B(1+t)$$

Put $t = -1$

Then -2 = A

Put t = -2

Then -4 = B(1 – 2)

-4 = -B

B = 4

$\therefore \quad I = \int \dfrac{-2}{1+t}dt + \int \dfrac{4}{2+t}dt$

$$= -2\log|1+t| + 4\log|2+t| + c$$

$$I = -2\log|1+\sin x| + 4\log|2+\sin x| + c$$

9. Soln. Let $I = \int \dfrac{\log x}{(x+1)^2}dx$

$$= \int \underset{I}{\log x}.\underset{II}{\frac{1}{(x+1)^2}}dx$$

Integrating by parts

$$I = \log x \int \frac{dx}{(x+1)^2} - \int\left(\frac{d}{dx}\log x \int \frac{dx}{(x+1)^2}\right)dx$$

$$\Rightarrow I = -\frac{\log x}{(x+1)} + \int \frac{1}{x(x+1)}dx$$

$$\left[\because \frac{d}{dx}\frac{1}{x^2} = -\frac{1}{x}\right]$$

$$= -\frac{\log x}{(x+1)} + \int \frac{(x+1)-x}{x(x+1)}dx$$

[Add and subtract x]

$$\Rightarrow I = -\frac{\log x}{(x+1)} + \int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$\Rightarrow I = -\frac{\log x}{(x+1)} + \log|x| - \log|x+1| + C$$

$\therefore \quad I = \log\left|\dfrac{x}{x+1}\right| - \dfrac{\log x}{(x+1)} + C.$

10. Soln. $\dfrac{\cos 2x - \cos 2x}{\cos x - \cos\alpha}$

$$= \frac{-2\sin\left(\frac{2x+2\alpha}{2}\right)\sin\left(\frac{2x-2\alpha}{2}\right)}{-2\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$= \frac{\sin(x+\alpha)\sin(x-\alpha)}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$=\frac{\left[\left\{2\sin\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x+\alpha}{2}\right)\right\}\left\{2\sin\left(\frac{x-\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)\right\}\right]}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$[\because \sin 2\theta = 2\sin\theta\cos\theta]$

$$=4\cos\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)$$

$$=2\left[\cos\left(\frac{x+\alpha}{2}+\frac{x-\alpha}{2}\right)+\cos\left(\frac{x+\alpha}{2}-\frac{x-\alpha}{2}\right)\right]$$

$[\because \cos(A+B)+\cos(A-B)=2\cos A\cos B]$

Now,

$$\int\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}=\int(2\cos x+2\cos\alpha)dx$$

$$=2\sin x+2x\cos\alpha+C$$

Ans.

11. Soln.

$$\int\frac{dx}{x(x^5+3)}=\int\frac{x^4}{x^5(x^5+3)}dx$$

Put, $(x^5+3)=t$

$\Rightarrow \quad 5x^4dx=dt$

$\Rightarrow \quad x^4dx=\frac{dt}{5}$

$$\therefore \int\frac{dx}{x(x^5+3)}=\frac{1}{5}\int\frac{dt}{t(t-3)}$$

Let $\frac{1}{t(t-3)}=\frac{A}{t}+\frac{B}{t-3}$

$1=A(t-3)+Bt$

$\Rightarrow \quad 1=At-3A+Bt$

$\Rightarrow \quad 1=(A+B)t-3A$

On comparing the co-efficient, we get

1 = -3A

$\Rightarrow \quad A=-1/3$

$A+B=0$

$\Rightarrow \quad B=1/3$

Now,

$$\frac{1}{5}\int\frac{dt}{t(t-3)}=\frac{1}{5}\int\left(\frac{-1}{3t}+\frac{1}{3(t-3)}\right)dt$$

$$=\frac{-1}{15}\log t+\frac{1}{15}\log(t-3)+C$$

$$=\frac{-1}{15}\left(\log\left(\frac{t}{t-3}\right)\right)+C$$

$[\because \log(a/b)=\log a-\log b]$

$$=\frac{-1}{15}\left(\log\frac{(x^5+3)}{(x^5+3-3)}\right)+C$$

$$=\frac{1}{15}\left(\log\frac{x^5}{x^5+3}\right)+C \qquad \text{Ans.}$$

12. Soln.

$$I=\int\left[\frac{(x-2)-2}{(x-2)^3}\right]e^xdx=\int\frac{e^x}{(x-2)^2}dx-2\int\frac{e^x}{(x-2)^3}dx$$

$$=\frac{e^x}{(x-2)^2}+2\int\frac{e^xdx}{(x-2)^3}-2\int\frac{e^xdx}{(x-2)^3}=\frac{e^x}{(x-2)^2}+C$$

13. Soln. Given,

$$\int e^x(\tan x+1)\sec x\,dx=e^x f(x)+C$$

$\Rightarrow \quad \int e^x(\tan x\sec x+\sec x)dx=e^xf(x)+C$

$\Rightarrow \quad \int e^x(\sec x+\tan x\sec x)\,dx=e^x f(x)+C$

$\Rightarrow \quad e^x\sec x+c=e^xf(x)+C$

$\Rightarrow \quad f(x)=\sec x$

[Note: $\int e^x[f(x)+f'(x)]dx=e^x.f(x)+C$, Here $f(x)=\sec x$]

14. Soln. Let

$$I=\int\frac{\tan\theta+\tan^3\theta}{1+\tan^3\theta}d\theta=\int\frac{\tan\theta(1+\tan^2\theta)}{1+\tan^3\theta}d\theta$$

$$=\int\frac{\tan\theta\sec^2\theta}{1+\tan^3\theta}d\theta$$

Let $\tan\theta=z \Rightarrow \sec^2\theta d\theta=dz$

$$\therefore \quad I=\int\frac{zdz}{1+z^3}=\int\frac{zdz}{(1+z)(z^2-z+1)}$$

Now $\frac{z}{(1+z)(z^2-z+1)}=\frac{A}{1+z}+\frac{Bz+C}{z^2-z+1}$

$\Rightarrow \quad z=A(z^2-z+1)+(Bz+C)(1+z)$

Putting z = -1, we get, $A=-\frac{1}{3}$

Putting z = 0, we get, $C=\frac{1}{3}$

Putting z = 1, we get, $B=\dfrac{1}{3}$

$$\therefore \frac{z}{(1+z)(z^2-z+1)}=\frac{-1}{3(1+z)}+\frac{\frac{1}{3}z+\frac{1}{3}}{z^2-z+1}$$

$$\Rightarrow I=-\frac{1}{3}\int\frac{dz}{1+z}+\frac{1}{3}\int\frac{z+1}{z^2-z+1}dz$$

$$=-\frac{1}{3}\log|1+z|+\frac{1}{3\times 2}\int\frac{2z-1+3}{z^2-z+1}dz$$

$$=-\frac{1}{3}\log|1+z|+\frac{1}{6}\int\frac{2z-1}{z^2-z+1}dz+\frac{1}{2}\int\frac{dz}{z^2-z+1}$$

$$I=-\frac{1}{3}\log|1+z|+\frac{1}{6}\log|z^2-z+1|+I_1+C \quad .....(i)$$

Where

$$I_1=\frac{1}{2}\int\frac{dz}{z^2-z+1}=\frac{1}{2}\int\frac{dz}{\left(z-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$=\frac{1}{2}\cdot\frac{2}{\sqrt{3}}+\tan^{-1}\left(\frac{z-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$\therefore I=-\frac{1}{3}\log|1+z|+\frac{1}{6}\log|z^2-z+1|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2z-1}{\sqrt{3}}\right)+C$$

Putting z = $\tan\theta$

$$\therefore I=-\frac{1}{3}\log|1+\tan\theta|+\frac{1}{6}\log|\tan^2\theta-\tan\theta+1|$$

$$+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta-1}{\sqrt{3}}\right)+C$$

15. Soln. Let $3x+5=A.\dfrac{d}{dx}(x^2-8x+7)+B$

$$\Rightarrow 3x+5=A(2x-8)+B$$

$$\Rightarrow \quad 3x+5=2Ax-8A+B$$

Equating the coefficient of x and the constant, we get

$$2A=3\,and\;-8A+B=5$$

$$\Rightarrow \quad A=\frac{3}{2}\,and-8\times\frac{3}{2}+B=5\Rightarrow B=5+12=17$$

Hence,

$$\int\frac{3x+5}{\sqrt{x^2-8x+7}}dx=\int\frac{\frac{3}{2}(2x-8)+17}{\sqrt{x^2-8x+7}}dx=\frac{3}{2}\int\frac{(2x-8)}{\sqrt{x^2-8x+7}}dx+17\int\frac{dx}{\sqrt{x^2-8x+7}}$$

$$=\frac{3}{2}I_1+17I_2 \quad ....(i)$$

Where

$$I_1=\int\frac{2x-8}{\sqrt{x^2-8x+7}}dx,\ I_2=\int\frac{dx}{\sqrt{x^2-8x+7}}$$

Now $I_1=\displaystyle\int\frac{2x-8}{\sqrt{x^2-8x+7}}dx$

Put $x^2-8x+7=z^2$

$$\Rightarrow (2x-8)dx=2z\,dz$$

$$\therefore \quad I_1=\int\frac{2zdz}{z}=2\int dz=2z+C_1$$

$$I_1=2\sqrt{x^2-8x+7}+C_1$$

.................(ii)

$$I_2=\int\frac{dx}{\sqrt{x^2-8x+7}}=\int\frac{dx}{\sqrt{x^2-2x.4+16-16+7}}=\int\frac{dx}{\sqrt{(x-4)^2-3^2}}$$

$$=\log\left|(x-4)+\sqrt{(x-4)^2-3^2}\right|+C_2$$

$$I_2=\log\left|(x-4)+\sqrt{x^2-8x+7}\right|+C_2$$

Putting the value of I_1 and I_2 in (i), we get

$$\int\frac{3x+5\,dx}{\sqrt{x^2-8x+7}}=\frac{3}{2}.\sqrt{x^2-8x+7}+17\log|(x-4)+\sqrt{x^2-8x+7}|+(C_1+C_2)$$

$$=3\sqrt{x^2-8x+7}+17\log|(x-4)+\sqrt{x^2-8x+7}|+C$$

$$\left[Note:\int\frac{dx}{\sqrt{x^2-a^2}}=\log|x+\sqrt{x^2-a^2}|+C\right]$$

16. Soln. The given integral is in he form of $\displaystyle\int\frac{(px+q)}{ax^2+bx+c}dx$.

Therefore, we express

$$(5x-2)=A\frac{d}{dx}(1+2x+3x^2)+B=A(2+6x)+B$$

Equating the coefficients of x and the constant term on both the sides, we get

6A = 5 and 2A + B = -2 or

$$A=\frac{5}{6}\,and\,B=-\frac{11}{3}.$$

$$\therefore \int\frac{(5x-2)}{(1+2x+3x^2)}dx=\frac{5}{6}\int\frac{2+6x}{(1+2x+3x^2)}dx-\frac{11}{3}\int\frac{dx}{3x^2+2x+1}$$

$$= \frac{5}{6} I_1 - \frac{11}{3} I_2, \qquad \text{(say)}$$

In I_1 putting $1+2x+3x^2 = t$, so that

$(2+6x)dx = dt$

$$\therefore\ I_1 = \int \frac{dt}{t} = \log|t| + C_1 = \log|3x^2+2x+1| + C_1$$

And

$$I_2 = \int \frac{dx}{3x^2+2x+1} = \int \frac{dx}{3\left(x^2+\frac{2x}{3}+\frac{1}{3}\right)} = \frac{1}{3}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2}$$

Putting $\left(x+\frac{1}{3}\right) = t$, so that dx = dt, we get

$$I_2 = \frac{1}{3}\int \frac{dt}{t^2 + \left(\frac{\sqrt{2}}{3}\right)^2} = \frac{1}{3.\frac{\sqrt{2}}{3}} \tan^{-1} \frac{t}{\frac{\sqrt{2}}{3}} + C_2$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{3t}{\sqrt{2}} + C_2 = \frac{1}{\sqrt{2}} \tan^{-1} \frac{3\left(x+\frac{1}{3}\right)}{\sqrt{2}} + C_2$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{(3x+1)}{\sqrt{2}} + C_2$$

Using (ii) and (iii) in (i), we get

$$\int \frac{(5x-2)}{(1+2x+3x^2)}.dx = \frac{5}{6}\log|3x^2+2x+1| - \frac{11}{3\sqrt{2}} \tan^{-1} \frac{(3x+1)}{\sqrt{2}} + C$$

Where, $\quad C = C_1 + C_2$

17. Soln. Let

$$I = \int e^x \frac{x^2+1}{(x+1)^2} dx = \int e^x \left(1 - \frac{2x}{(x+1)^2}\right) dx = \int e^x - 2\left(\frac{e^x x.dx}{(x+1)^2}\right)$$

$$= e^x - 2\int e^x \frac{x+1-1}{(x+1)^2} dx = e^x - 2\int e^x \left[\frac{1}{x+1} + \frac{-1}{(x+1)^2}\right] dx$$

$$= e^x - 2e^x.\frac{1}{x+1} + C$$

[Note: $\int e^x \{f(x) + f'(x)\}dx = e^x.f(x) + C$]

18. Soln. Let $\quad I = \int \frac{\sin(x-a)}{\sin(x+a)} dx$

Let $x + a = t \Rightarrow x = t - a \quad \Rightarrow \quad dx = dt$

$$\therefore\ I = \int \frac{\sin(t-2a)}{\sin t} dt = \int \frac{\sin t.\cos 2a - \cos t.\sin 2a}{\sin t} dt$$

$$= \cos 2a \int dt - \int \sin 2a.\cot dt = \cos 2a.t - \sin 2a.\log|\sin t| + C$$

$$= \cos 2a.(x+a) - \sin 2a.\log|\sin(x+a)| + C$$

$$= x\cos 2a + a\cos 2a - (\sin 2a)\log|\sin(x+a)| + C$$

19. Soln. Let $\quad I = \int x \sin^{-1} x dx$

$$= \sin^{-1} x.\frac{x^2}{2} - \int \frac{x^2}{2\sqrt{1-x^2}} dx \quad \text{[By using integrating by part]}$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}} dx = \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \sqrt{1-x^2}\, dx - \frac{1}{2}\int \frac{dx}{\sqrt{1-x^2}}$$

$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\sin^{-1} x + \frac{1}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right] + C$$

$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + \frac{1}{4}\sin^{-1} x + C$$

$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{4}\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + C$$

20. Soln. Let $I = \int e^x \left(\frac{\sin 4x - 4}{1 - \cos 4x}\right) dx$

$$= \int e^x \left(\frac{2\sin 2x.\cos 2x - 4}{2\sin^2 2x}\right) dx$$

$\left[\because\ \sin 2x = 2\sin x.\cos x \text{ and } \cos 2x = 1 - 2\sin^2 x\right]$

$$= \int e^x (\cot 2x - 2\operatorname{cosec}^2 2x) dx$$

Let

$f(x) = \cot 2x \qquad \therefore\ f'(x) = -2\operatorname{cosec}^2 2x$

$$\therefore \qquad I = \int e^x (f(x) + f'(x)) dx$$

$$\Rightarrow\ I = e^x.f(x) + C = e^x.\cot 2x + C$$

$\left[\because \int e^x (f(x) + f'(x))dx = e^x f(x) + C\right]$

21. Soln. We can express the N^r as

$$5x+3 = A\frac{d}{dx}(x^2+4x+10) + B$$

$$\Rightarrow\ 5x+3 = A(2x+4) + B \Rightarrow 5x+3 = 2Ax + (4A+B)$$

Equating the coefficients, we get

$$2A = 5 \text{ and } 4A + B = 3$$

$$A = \frac{5}{2} \quad \Rightarrow \quad 4\times\frac{5}{2} + B = 3 \quad \Rightarrow B = 3 - 10 = -7$$

$$\therefore \qquad 5x+3 = \frac{5}{2}(2x+4) + (-7)$$

$$\therefore\ I=\int\frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}dx=\frac{5}{2}\int\frac{(2x+4)}{\sqrt{x^2+4x+10}}dx-7\int\frac{dx}{\sqrt{x^2+4x+10}}$$

$$I=\frac{5}{2}I_1-7I_2$$

Where

$$I_1=\int\frac{2x+4}{\sqrt{x^2+4x+10}}dx \text{ and } I_2=\int\frac{dx}{\sqrt{x^2+4x+10}}$$

Now, $$I_1=\int\frac{(2x+4)}{\sqrt{x^2+4x+10}}dx$$

Let $x^2+4x+10=t \Rightarrow (2x+4)\,dx=dt$

$$\therefore\quad I_1=\int\frac{dt}{\sqrt{t}}=\int t^{-1/2}dt=\frac{t^{-1/2+1}}{-\frac{1}{2}+1}+C_1=2\sqrt{t}+C_1$$

$$I_1=2\sqrt{x^2+4x+10}+C_1$$

Again,

$$I_2=\int\frac{dx}{\sqrt{x^2 2.x.2+2^2-4+10}}=\int\frac{dx}{\sqrt{(x+2)^2+\left(\sqrt{6}\right)^2}}$$

$$=\log|(x+2)+\sqrt{x^2+4x+10}|+C_2$$

Putting the value of I_1 and I_2 in (i), we get

$$I=\frac{5}{2}\times2\sqrt{x^2+4x+10}-7\log|(x+2)+\sqrt{x^2+4x+10}|+\left(\frac{5}{2}C_1-7C_2\right)$$

$$=5\sqrt{x^2+4x+10}-7\log|(x+2)+\sqrt{x^2+4x+10}|+C$$

22. Soln. Let $$I=\int\frac{x^2}{(x^2+4)(x^2+9)}dx$$

Put $x^2=t$, we get

$$\therefore\quad\frac{x^2}{(x^2+4)(x^2+9)}=\frac{t}{(t+4)(t+9)}$$

Now, $$\frac{t}{(t+4)(t+9)}=\frac{A}{t+4}+\frac{B}{t+9}=\frac{A(t+9)+B(t+4)}{(t+4)(t+9)}$$

$$\Rightarrow\quad t=(A+B)t+(9A+4B)$$

Equating the coefficients, we get

$$A+B=1,\qquad 9A+4B=0$$

Solving above two equations, we get

$$A=-\frac{4}{5},\ B=\frac{9}{5}$$

$$\therefore\quad\frac{x^2}{(x^2+4)(x^2+9)}=-\frac{4}{5(x^2+4)}+\frac{9}{5(x^2+9)}$$

$$I=-\frac{4}{5}\int\frac{dx}{x^2+2^2}+\frac{9}{5}\int\frac{dx}{x^2+3^2}$$

$$=-\frac{4}{5}\times\frac{1}{2}\tan^{-1}\frac{x}{2}+\frac{9}{5}\times\frac{1}{3}\tan^{-1}\frac{x}{3}+C$$

$$=-\frac{2}{5}\tan^{-1}\frac{x}{2}+\frac{3}{5}\tan^{-1}\frac{x}{3}+C$$

23. Soln. Let $$I=\int\left(\sqrt{\cot x}+\sqrt{\tan x}\right)dx$$

$$I=\int\left(\frac{\sqrt{\cos x}}{\sqrt{\sin x}}+\frac{\sqrt{\sin x}}{\sqrt{\cos x}}\right)dx=\int\frac{(\cos x+\sin x)}{\sqrt{\sin x.\cos x}}dx$$

Let $\sin x-\cos x=t\Rightarrow\qquad(\cos x+\sin x)\,dx=dt$

Also $\because\ \sin x-\cos x=t$

$$\Rightarrow\qquad(\sin x-\cos x)^2=t^2$$

$$\Rightarrow\qquad\sin^2x+\cos^2x-2\sin x.\cos x=t^2$$

$$\Rightarrow\qquad1-2\sin x.\cos x=t^2$$

$$\Rightarrow\qquad\sin x.\cos x=\frac{1-t^2}{2}$$

Therefore, $$I=\int\frac{dt}{\sqrt{\frac{1-t^2}{2}}}=\sqrt{2}\int\frac{dt}{\sqrt{1-t^2}}$$

$$=\sqrt{2}\sin^{-1}t+C=\sqrt{2}\sin^{-1}(\sin x-\cos x)+C$$

24. Soln. Let $$I=\int\frac{x+2}{(x^2+3x+3)\sqrt{x+1}}dx$$

Putting $x+1=t^2\qquad\Rightarrow\qquad dx=2t\ dt$

$$I=2\int\frac{t^2+1}{t^4+t^2+1}dt=2\int\frac{1+\left(\frac{1}{t}\right)^2}{t^2+\frac{1}{t^2}+1}dt$$

[Dividing N^r and D^r by t^2]

$$=2\int\frac{1+\left(\frac{1}{t}\right)^2}{\left(t-\frac{1}{t}\right)^2+3}dt$$

Now, Put

$$t-\frac{1}{t}=z\qquad\Rightarrow\qquad\left(1+\frac{1}{t^2}\right)dt=dz\text{, we get}$$

$$I = 2\int \frac{dz}{z^2+3} = \frac{2}{\sqrt{3}}\tan^{-1}\frac{z}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{3}}\right) + C = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{3}t}\right) + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3(x+1)}}\right) + C$$

25. Soln. Let $I = \int 5^{5^{5^x}}.5^{5^x}.5^x.dx$

Putting

$$5^x = t \Rightarrow 5^x.\log 5 dx = dt \ or \ 5^x.dx = \frac{dt}{(\log 5)}$$

Therefore,

$$I = \int 5^{5^{5^x}}.5^{5^x}.5^x.dx = \int 5^{5^t}.5^t.\frac{dt}{(\log 5)} = \frac{1}{(\log 5)}\int 5^{5^t}.5^t.dt$$

Again, putting $5^t = u$, $5^t dt = \dfrac{du}{(\log 5)}$

Therefore, $I = \dfrac{1}{(\log 5)}\displaystyle\int 5^u.\frac{du}{(\log 5)}$

$$= \frac{1}{(\log 5)^2}\int 5^u.\frac{du}{(\log 5)^2.(\log 5)} + C$$

$$= \frac{5^u}{(\log 5)^3} + C = \frac{5^{5^t}}{(\log 5)^3} + C = \frac{5^{5^{5^x}}}{(\log 5)^3} + C$$

26. Soln. Let $I = \int \sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}dx$

Putting

$\sqrt{x} = \cos\theta$, *i.e.*, $x = \cos^2\theta \Rightarrow \theta = \cos^{-1}\sqrt{x}$ and $dx = -2\cos\theta\sin\theta d\theta$, we get

$$\therefore \ I = \int\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-2\sin\theta\cos\theta)\,d\theta$$

$$= -2\int\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}(\sin\theta\cos\theta)\,d\theta = -2\int\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\theta\right)d\theta$$

$$= -2\int 2\sin^2\frac{\theta}{2}\cos\theta d\theta = -2\int(1-\cos\theta)\cos\theta d\theta$$

$$= -2\int(1-\cos\theta)\cos\theta.\,d\theta = -2\sin\theta + \int(1+\cos 2\theta).\,d\theta$$

$$= -2\sin\theta + \int 1.d\theta + \int\cos 2\theta.d\theta = -2\sin\theta + \theta + \frac{\sin 2\theta}{2} + C$$

$$= -2\sqrt{1-\cos^2\theta} + \theta + \frac{2\sqrt{1-\cos^2\theta}.\cos\theta}{2} + C$$

$$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x} + C$$

27. Soln. Let

$$I = \int\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}dx, x \in [0,1]$$

We know that $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \dfrac{\pi}{2}$

$$\Rightarrow \ \sin^{-1}\sqrt{x} = \frac{\pi}{2} - \cos^{-1}\sqrt{x}$$

$$\therefore \ I = \int\frac{\frac{\pi}{2} - 2\cos^{-1}\sqrt{x}}{\pi/2}dx$$

$$= \int 1.dx - \frac{4}{\pi}\int 1.\cos^{-1}\sqrt{x}dx$$

$$= x - \frac{4}{\pi}\left[x.\cos^{-1}\sqrt{x} - \int x.\frac{-1}{\sqrt{1-x}}.\frac{1}{2\sqrt{x}}dx\right] + C$$

$$= x - \frac{4}{\pi}\left[x\cos^{-1}\sqrt{x} + \frac{1}{2}\int\sqrt{\frac{x}{1-x}}dx\right] + C$$

Put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta d\theta$

$$\therefore \ I = x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}\int\sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}.2\sin\theta\cos\theta d\theta + C$$

$$= x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}\int\frac{\sin\theta}{\cos\theta}.2\sin\theta\cos\theta d\theta + C$$

$$= x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}\int(1-\cos 2\theta)\,d\theta + C$$

$$= x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}\left[\theta - \frac{\sin 2\theta}{2}\right] + C$$

$$= x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}[\theta - \sin\theta\cos\theta] + C$$

$$= x - \frac{4}{\pi}\times\cos^{-1}\sqrt{x} - \frac{2}{\pi}\left[\sin^{-1}\sqrt{x} - \sqrt{x}\sqrt{1-x}\right] + C$$

28. Soln. Let $I = \int\dfrac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$

$$= \sin^{-1}x\int\frac{x}{\sqrt{1-x^2}}dx-\int\left[\frac{d}{dx}(\sin^{-1}x)\int\frac{x}{\sqrt{1-x^2}}dx\right]dx$$

(Applying integration by parts)

Firstly, let us evaluate the integral $\int\frac{x}{\sqrt{1-x^2}}dx$

Put $t=1-x^2 \Rightarrow dt=-2x\ dx.$

$$\therefore\ \int\frac{x}{\sqrt{1-x^2}}dx=-\frac{1}{2}\int\frac{dt}{\sqrt{t}}=-\sqrt{t}=-\sqrt{1-x^2}$$

$$\therefore\ I=\sin^{-1}x\left(-\sqrt{1-x^2}\right)-\int\frac{1}{\sqrt{1-x^2}}\left(-\sqrt{1-x^2}\right)dx$$

$$=-\sqrt{1-x^2}\sin^{-1}x+\int dx=-\sqrt{1-x^2}\sin^{-1}x+x+C$$

29. Soln. Let $I=\int\frac{x^2-3x+1}{\sqrt{1-x^2}}dx$

$$=-\int\frac{\left(-x^2+3x-1\right)}{\sqrt{1-x^2}}dx=-\int\frac{\left(1-x^2\right)+3x-2}{\sqrt{1-x^2}}dx$$

i.e., $I=-\int\sqrt{1-x^2}dx+\int\frac{-3x+2}{\sqrt{1-x^2}}dx$

$$=-\int\sqrt{1-x^2}dx+\frac{3}{2}\int\frac{-2x}{\sqrt{1-x^2}}dx+2\int\frac{1}{\sqrt{1-x^2}}dx$$

$$=-\int\sqrt{1-x^2}dx+\frac{3\times2}{2}\sqrt{1-x^2}+2\sin^{-1}x+C$$

$$=-\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]+3\sqrt{1-x^2}+2\sin^{-1}x+C$$

$$=-\frac{x}{2}\sqrt{1-x^2}+\frac{3}{2}\sin^{-1}x+3\sqrt{1-x^2}+C$$

30. Soln. Let $I=\int e^{2x}\sin(3x+1)dx$

On integrating by parts, taking e^x as first function, we have

$$=e^{2x}\int\sin(3x+1)dx-\int\left(\frac{d(e^{2x})}{dx}.\int\sin(3x+1)dx\right)dx$$

$$=e^{2x}\frac{[-\cos(3x+1)]}{3}-\int2e^{2x}.\frac{[-\cos(3x+1)]}{3}dx$$

$$=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{3}\int e^{2x}\cos(3x+1)dx$$

$$=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{3}\left[e^{2x}\int\cos(3x+1)dx-\int\left(\frac{d}{dx}(e^{2x}).\int\cos(3x+1)dx\right)dx\right]$$

$$=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{9}e^{2x}\sin(3x+1)-\frac{4}{9}\int e^{2x}\sin(3x+1)dx$$

$$=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{9}e^{2x}\sin(3x+1)-\frac{4}{9}I+C_1$$

$$\Rightarrow I+\frac{4}{9}I=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{9}e^{2x}\sin(3x+1)+C_1$$

$$\Rightarrow \frac{13I}{9}=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{9}e^{2x}\sin(3x+1)+C_1$$

$$\Rightarrow I=\frac{9}{13}\left[\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{9}e^{2x}\sin(3x+1)+C_1\right]$$

$$=\frac{9}{13}e^{2x}\left[\frac{2\sin(3x+1)-3\cos(3x+1)}{9}\right]+\frac{9}{13}C_1$$

$$=\frac{1}{13}e^{2x}\left[2\sin(3x+1)-3\cos(3x+1)\right]+C\left(where, C=\frac{9C_1}{13}\right)$$

SURE SHOT QUESTIONS

ARVIND ACADEMY

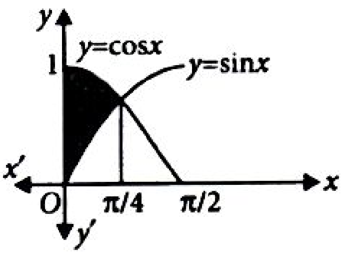
Chapter – 08 (Solution)

Application of Integrals

➤ MCQ [1 Marks]

1. Soln. (c): We have, $y = \cos x,\ 0 \le x \le \frac{\pi}{2}$ and

$y = \sin x,\ 0 \le x \le \frac{\pi}{2}$



Curve y = sinx and y = cosx intersect at $x = \pi/4$

∴ Required area

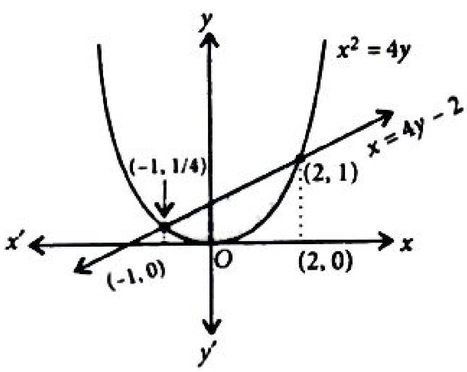
$$= \int_0^{\pi/4} \cos x\, dx - \int_0^{\pi/4} \sin x\, dx$$

$$= [\sin x]_0^{\pi/4} - [-\cos x]_0^{\pi/4}$$

$$= \left(\frac{1}{\sqrt{2}} - 0\right) + \left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= \frac{2}{\sqrt{2}} - 1 = (\sqrt{2} - 1) \text{ sq. unit}$$

2. Soln. (d): We have, $x^2 = 4y$ ....(i), a parabola with vertex (0, 0) and x = 4y – 2 ....(ii), a straight line.



Solving curve (i) and (ii), we get x = 2 and x = -1

∴ Required area = area of shaded region

$$= \int_{-1}^{2} \left[\left(\frac{x+2}{4}\right) - \frac{x^2}{4}\right] dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{1}{4}\left[2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}\right]$$

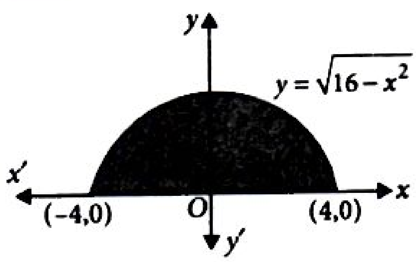
$$= \frac{1}{4} \times \frac{9}{2} = \frac{9}{8}\ sq.\ units$$

3. Soln. (a): We have, $y = \sqrt{16 - x^2}$

$\Rightarrow y^2 = 16 - x^2$

$\Rightarrow x^2 + y^2 = 4^2$, *a circle with centre* $(0, 0)$

And radius 4 units.

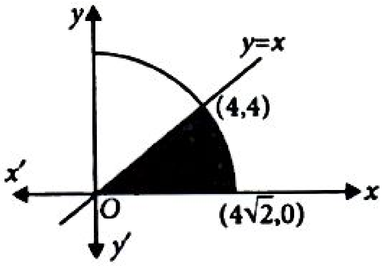


∴ Required area = area of shaded region

$$= \int_{-4}^{4} \sqrt{4^2 - x^2}\, dx = \left[\frac{x}{2}\sqrt{4^2 - x^2} + \frac{4^2}{2}\sin^{-1}\frac{x}{4}\right]_{-4}^{4}$$

$$= [8\sin^{-1}(1) - 8\sin^{-1}(-1)] = \frac{8\pi}{2} + \frac{8\pi}{2} = 8\pi\ sq.units$$

4. Soln. (b): We have, $x^2 + y^2 = 32$ ......(i), a circle with centre (0, 0) and radius $4\sqrt{2}$, and y = x .....(ii), a straight line Solving (i) and (ii), we get point of intersection (4, 4).



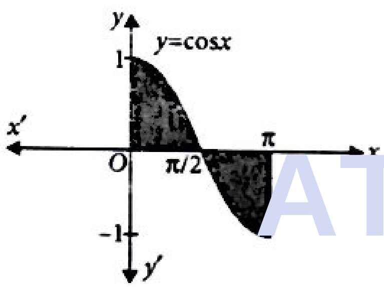
$$\therefore \text{ Required area} = \int_0^4 x\,dx + \int_0^{4\sqrt{2}} \sqrt{32-x^2}\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^4 + \left[\frac{x}{2}\sqrt{32-x^2} + \frac{32}{2}\sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$$= 8 + \left[16\times\frac{\pi}{2}\right] - \left[2\sqrt{32-4^2} + 16\sin^{-1}\frac{1}{\sqrt{2}}\right]$$

$$= 8 + 8\pi - 8 - 4\pi = 4\pi \ \ sq.\ units$$

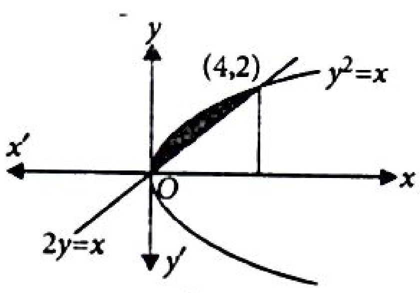
5. Soln. (a): We have, $y = \cos x$



∴ Required area = area of shaded region

$$= 2\int_0^{\pi/2} \cos dx = 2[\sin x]_0^{\pi/2} = 2\,sq.units$$

6. Soln. (a): We have 2y = x ....(i), a straight line, and $y^2 = x$ ....(ii), a parabola with vertex (0, 0).



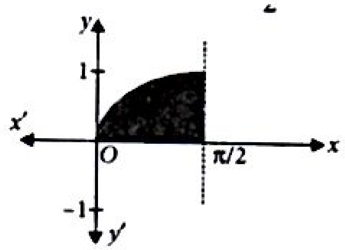
Solving (i) and (ii), we get x = 0 and x = 4.

∴ Required area = area of shaded region

$$= \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right)dx = \left[\frac{2}{3}x^{3/2} - \frac{x^2}{4}\right]_0^4 = \frac{2}{3}\times 8 - \frac{16}{4}$$

$$= \frac{16-12}{3} = \frac{4}{3}\ sq.\ units$$

7. Soln. (d): We have, $y = \sin x,\ 0 \le x \le \frac{\pi}{2}$



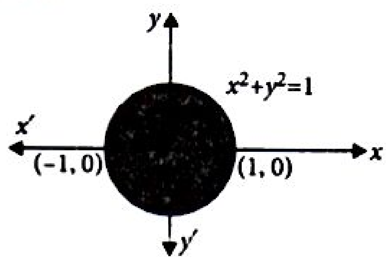
∴ Required area = area of shaded region

$$= \int_0^{\pi/2} \sin x dx = [-\cos x]_0^{\pi/2} = -[0-1] = 1\,sq.unit$$

8. Soln. (a): Area of the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab sq. units

∴ Required area = $\pi \times 5 \times 4 = 20\pi$ sq. units

9. Soln. (b): We have, $x^2 + y^2 = 1$, a circle with centre (0, 0) and radius 1.



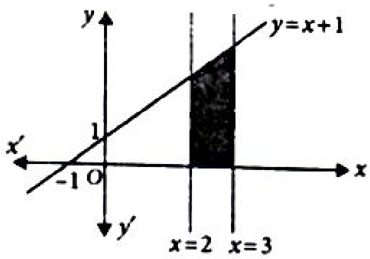
∴ Required area = area of shaded region

$$= 4\int_0^1 \sqrt{1-x^2}\,dx = 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$$

$$= 4\times\frac{1}{2}\times\frac{\pi}{2} = \pi \ \ sq.units$$

10. Soln. (a): We have, $y = x + 1$, a straight line

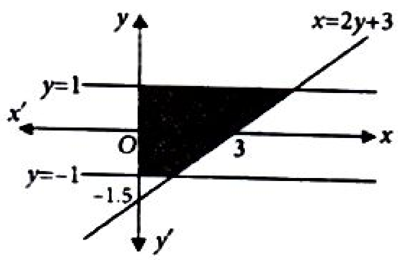
∴ Required area = area of shaded region



$$=\int_{2}^{3}(x+1)dx=\left[\frac{x^2}{2}+x\right]_2^3=\left(\frac{9}{2}+3\right)-\left(\frac{4}{2}+2\right)$$

$$=\frac{15}{2}-4=\frac{7}{2}\ sq.units$$

11. Soln. (c): We have $x=2y+3$, a straight line



$\therefore$ Required area = area of shaded region

$$=\int_{-1}^{1}(2y+3)dy=\left[y^2+3y\right]_{-1}^{1}$$

$$=(1+3)-(1-3)=4+2=6\ sq.units$$

➢ Assertion-Reasoning (1 mark)

12. **Sol. (a)** Both A and R are true and R is the correct explanation of A.

Explanation: Both A and R are true and R is the correct explanation of A.

13. **Sol. (b)** Both A and R are true but R is not the correct explanation of A.

Explanation: Both A and R are true but R is not the correct explanation of A.

14. **Sol. (d)** A is false but R is true.

Explanation: $A=2\int_0^4 x dy$

$=2\int_0^4 \sqrt{y}dy$

$=2\times\frac{2}{3}\left[y^{3/2}\right]_0^4$

$=\frac{4}{3}\times 8$

$=\frac{32}{3}$

15. **Sol. (a)** Both A and R are true and R is the correct explanation of A.

Explanation: Both A and R are true and R is the correct explanation of A.

16. **Sol. (a)** Both A and R are true and R is the correct explanation of A.
Explanation: Both A and R are true and R is the correct explanation of A.

➢ Case Study Question

17. **Sol.**

(i) **(c)** $x^2 = -250y$

Explanation: $x^2 = -250y$

(ii) **(c)** $\frac{1000}{3}$

Explanation: $\frac{1000}{3}$

(iii) **(b)** Even

Explanation: Even

(iv) **(a)** $\frac{1000}{3}$

Explanation: $\frac{1000}{3}$

(v) **(d)** None of these

Explanation: None of these

18. **Sol.**

(i) **(c)** $y=\frac{3}{2}(x+1)$

Explanation: Equation of line AB is
$y-0=\frac{3-0}{1+1}(x+1)\Rightarrow y=\frac{3}{2}(x+1)$

(ii) **(d)** $y=\frac{-1}{2}x+\frac{7}{2}$

Explanation: Equation of line BC is $y-3=\frac{2-3}{3-1}(x-1)$
$\Rightarrow y=-\frac{1}{2}x+\frac{1}{2}+3\Rightarrow y=\frac{-1}{2}x+\frac{7}{2}$

(iii) **(a)** 8 sq. units

Explanation: Area of region ABCD
= Area of ΔABE + Area of region BCDE
$=\int_{-1}^{1}\frac{3}{2}(x+1)dx+\int_1^3\left(\frac{-1}{2}x+\frac{7}{2}\right)dx$
$=\frac{3}{2}\left[\frac{x^2}{2}+x\right]_{-1}^{1}+\left[\frac{-x^2}{4}+\frac{7}{2}x\right]_1^3$
$=\frac{3}{2}\left[\frac{1}{2}+1-\frac{1}{2}+1\right]+\left[\frac{-9}{4}+\frac{21}{2}+\frac{1}{4}-\frac{7}{2}\right]$
$=3+5=8$ sq. units

(iv) (a) 4 sq. units

Explanation: Equation of line AC is y - 0 = $\frac{2-0}{3+1}(x+1)$

$\Rightarrow$ y = $\frac{1}{2}(x+1)$

$\therefore$ Area of $\triangle$ADC = $\int_{-1}^{3}\frac{1}{2}(x+1)dx = \left[\frac{x^2}{4}+\frac{1}{2}x\right]_{-1}^{3}$

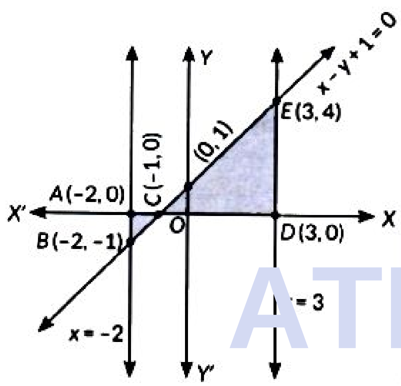
= $\frac{9}{4}+\frac{3}{2}-\frac{1}{4}+\frac{1}{2}$ = 4 sq. units

(v) (b) 4 sq. units

Explanation: Area of ΔABC = Area of region ΔABCD - Area of ACD = 8 - 4 = 4 sq. units.

Questions

19. Soln.



Required area = ar(ΔABC) + ar(ΔCDE)

$$=\left|\int_{-2}^{-1}(x+1)\,dx\right|+\int_{-1}^{3}(x+1)\,dx=\left|\left[\frac{x^2}{2}+x\right]_{-2}^{-1}\right|+\left[\frac{x^2}{2}+x\right]_{-1}^{3}$$

$$=\left|\frac{1}{2}-1-(2-1)\right|+\left\{\frac{9}{2}+3-\left(\frac{1}{2}-1\right)\right\}=\frac{1}{2}+8$$

$$=\frac{17}{2}=8.5\,sq.units$$

20. Soln.

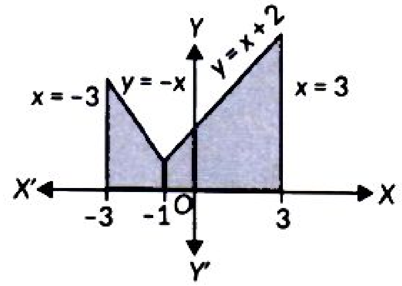
Here, $y=|x+1|+1$

$$y=\begin{cases}x+2 & if \quad x\geq -1\\ -x & if \quad x<-1\end{cases}$$

We know draw the lines: y = 0, x = 3, x = -3 and y = x + 2 if $x\geq -1$ ....(i)

$y=-x$ if $x<-1$ ....(ii)

Lines (i) and (ii) intersect at (-1, 1)



$\therefore$ Required area = $\int_{-3}^{-1}(-x)dx+\int_{-1}^{3}(x+2)\,dx$

$$=-\left[\frac{x^2}{2}\right]_{-3}^{-1}+\left[\frac{x^2}{2}+2x\right]_{-1}^{3}$$

$$=-\frac{1}{2}(1-9)+\frac{1}{2}(9-1)+2(3+1)$$

$$=4+4+8=16\,\text{sq. units}.$$

21. Soln. Equations of the given circles are,

$$x^2+y^2=4 \quad ....(i)$$

$$(x-2)^2+y^2=4 \quad ....(ii)$$

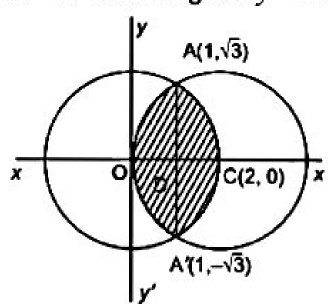
Equation (i) is a circle with centre O at the origin and radius 2. Equation (ii) is a circle with centre C(2, 0) and radius 2.

Solving equation (i) and (ii) we have

$$(x-2)^2+y^2=x^2+y^2$$

$$or \quad x^2-4x+4+y^2=x^2+y^2$$

Or x = 1 which gives $y=\pm\sqrt{3}$



Thus, the points of intersection of the given circles are $A(1,\sqrt{3})$ and $A'(1,-\sqrt{3})$

Required area of the enclosed region OACA'O between circles

= 2 [area of region ODCAO]

= 2 [are of region ODAO + area of region OCAD]

$$=2\left[\int_0^1 ydx+\int_1^2 ydx\right]$$

$$= 2\left[\int_0^1 \sqrt{4-(x-2)^2}\,dx + \int_1^2 \sqrt{4-x^2}\,dx\right] \quad [\text{from (i)}]$$

$$= 2\left[\frac{1}{2}(x-2)\sqrt{4-(x-2)^2} + \frac{1}{2}\times 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1$$

$$+2\left[\frac{1}{2}x\sqrt{4-x^2} + \frac{1}{2}\times 4\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \left[(x-2)\sqrt{4-(x-2)^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1$$

$$+\left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \left[-\sqrt{3} + 4\sin^{-1}\left(\frac{-1}{2}\right) - 4\sin^{-1}(-1)\right]$$

$$+\left[4\sin^{-1}1 - \sqrt{3} - 4\sin^{-1}\frac{1}{2}\right]$$

$$= \left[\left(-\sqrt{3} - 4\times\frac{\pi}{6}\right) + 4\times\frac{\pi}{2}\right] + \left[4\times\frac{\pi}{2} - \sqrt{3} - 4\times\frac{\pi}{6}\right]$$

$$= \frac{8\pi}{3} - 2\sqrt{3}$$

22. Soln. Given curves is

$$x^2 + y^2 = 32 \quad \ldots\ldots\text{(i)}$$

$$x^2 + y^2 = \left(\sqrt{32}\right)^2 = \left(4\sqrt{2}\right)^2$$

It is a circle with centre (0, 0) and radius $4\sqrt{2}$.

Given line is y = x ....(ii)

Solving equations (i) and (ii) for points of intersections,

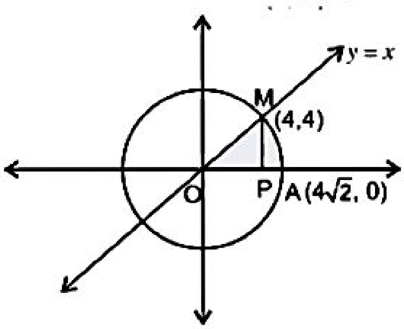
$$x^2 + x^2 = 32 \quad [\text{using (ii) in (i)}]$$

$$\Rightarrow \quad x^2 = 16$$

$$\Rightarrow \quad x = \pm 4$$

$$\Rightarrow \quad x = 4 \quad (\text{first quadrant})$$

$\therefore$ Point of intersection is (4, 4)



Required area = Area OMA

= Area OMP + Area MPA

$$= \int_0^4 x.dx + \int_4^{4\sqrt{2}} \sqrt{\left(4\sqrt{2}\right)^2 - x^2}.dx$$

$$= \left[\frac{x^2}{2}\right]_0^4 + \left[\frac{x}{2}\sqrt{\left(4\sqrt{2}\right)^2 - x^2} + \frac{\left(4\sqrt{2}\right)^2}{2}\sin^{-1}\left(\frac{x}{4\sqrt{2}}\right)\right]_4^{4\sqrt{2}}$$

$$= \frac{16}{2} + \left[\left\{\frac{4\sqrt{2}}{2}\sqrt{\left(4\sqrt{2}\right)^2 - \left(4\sqrt{2}\right)^2} + \frac{32}{2}\sin^{-1}1\right\} - \left\{\frac{4}{2}\sqrt{\left(4\sqrt{2}\right)^2 - (4)^2} + \frac{32}{2}\sin^{-1}\frac{1}{\sqrt{2}}\right\}\right]$$

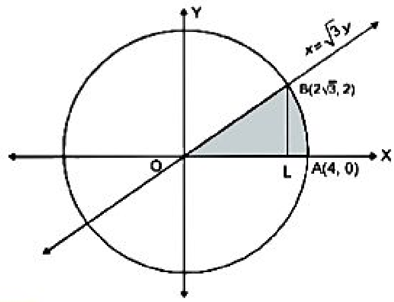
$$= 8 + \left(2\sqrt{2}(0) + 16\times\frac{\pi}{2}\right) - \left(2\times 4 + 16\times\frac{\pi}{4}\right)$$

$$= 8 + 8\pi - 8 - 4\pi$$

$= 4\pi$ *sq units* Ans.

23. Soln. Given, $x = \sqrt{3}y$

And $x^2 + y^2 = 16$,



$$\Rightarrow \quad \left(\sqrt{3}y\right)^2 + y^2 = 16$$

$$\Rightarrow \quad 4y^2 = 16$$

$$\Rightarrow \quad y^2 = 4$$

$$\Rightarrow \quad y = 2$$

$$\therefore \quad x = \sqrt{3}y = 2\sqrt{3}$$

$\therefore$ $B(2\sqrt{3}, 2)$ is the point of intersection in first quadrant.

Required area = Area under OBL + Area under LBA

$$= \int_0^{2\sqrt{3}} \frac{x}{\sqrt{3}}dx + \int_{2\sqrt{3}}^4 \sqrt{16 - x^2}\,dx$$

$$= \frac{1}{\sqrt{3}}\left(\frac{x^2}{2}\right)_0^{2\sqrt{3}} + \left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_{2\sqrt{3}}^4$$

$$= \frac{1}{2\sqrt{3}}(12 - 0) + \left(0 + 8\sin^{-1}1\right) - \left(\frac{2\sqrt{3}}{2}\sqrt{16-12} + 8\sin^{-1}\frac{2\sqrt{3}}{4}\right)$$

$$= \frac{6}{\sqrt{3}} + 8\times\frac{\pi}{2} - 2\sqrt{3} - 8\sin^{-1}\frac{\sqrt{3}}{2}$$

$$= \frac{6\sqrt{3}}{3} + 4\pi - 2\sqrt{3} - 8\times\frac{\pi}{3}$$

$$= 2\sqrt{3} + \frac{4\pi}{3} - 2\sqrt{3}$$

$= \dfrac{4\pi}{3}$ *sq. units.* Ans.

24. Soln. Given equations are

$y = 2x+1$ ....(i)

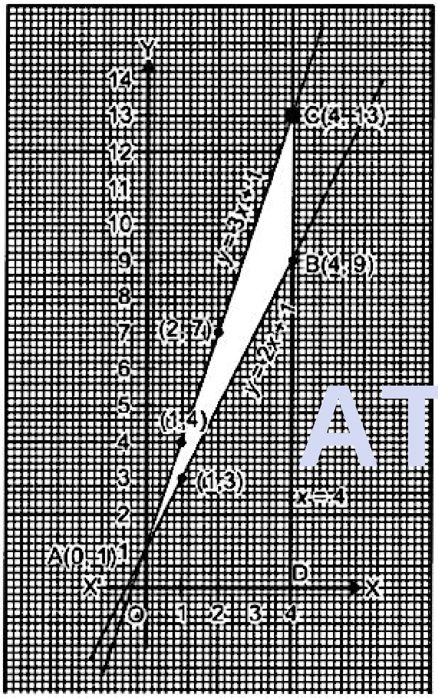
$y = 3x+1$ ....(ii)

Table for line (i)

X	0	1	2	4
Y	1	3	5	9

Table for line (ii),

X	0	1	2	4
Y	1	4	7	13



Area of triangular region ABC

= Area of the region OACDO – Area of the region OABDO

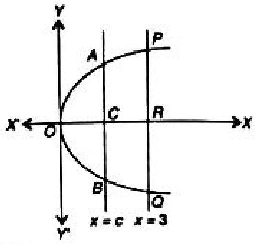
$= \int_0^4 [y \text{ line(ii)} - y \text{ line(i)}]dx$

$= \int_0^4 [(3x+1)-(2x+1)]dx$

$= \int_0^4 (3x+1-2x-1) = \int_0^4 x.dx$

$= \left[\frac{x^2}{2}\right]_0^4 = \left[\frac{(4)^2}{2}\right] = 8 \text{ sq. units}$

25. Soln.



Given,

Area OACBO = Area of APRQBCA

Or Area OACBO = $2\int_0^c y\,dx = 2\int_0^c \sqrt{x}dx$

$= 2\left[\frac{x^{3/2}}{3/2}\right]_0^c = \frac{4}{3}c^{3/2}$

Area of APRQBCA = $2\int_c^3 ydx = 2\int_c^3 x^{1/2}dx$

$= 2\left[\frac{x^{3/2}}{3/2}\right]_c^3 = \frac{4}{3}\left[3^{3/2} - c^{3/2}\right]$

$\therefore \quad \frac{4}{3}c^{3/2} = \frac{4}{3}\left[3^{3/2} - c^{3/2}\right]$

or $\quad c^{3/2} = 3^{3/2} - c^{3/2}$

or $\quad 2c^{3/2} = 3^{3/2} \text{ or } 4c^3 = 3^3$

or $\quad c = \sqrt[3]{27/4} = \frac{3}{2}\sqrt[3]{2}.$

26. Soln.

$\{(x,y): x^2+y^2 \le 2ax,\ y^2 \ge ax,\ x,y \ge 0\}$

Considering the inequations as equations

$x^2 + y^2 = 2ax$ ....(i)

Or $\quad x^2 - 2ax + y^2 = 0$

Or $\quad x^2 - 2ax + a^2 + y^2 = a^2$

Or $\quad (x-a)^2 + y^2 = a^2$ ....(ii)

It represents a circle whose

Centre is (a, 0) and radius r = a

$y^2 = ax$ ....(iii)

Vertex (0, 0)

Axis along x-axis.

Point of intersection, from (i) and (iii)

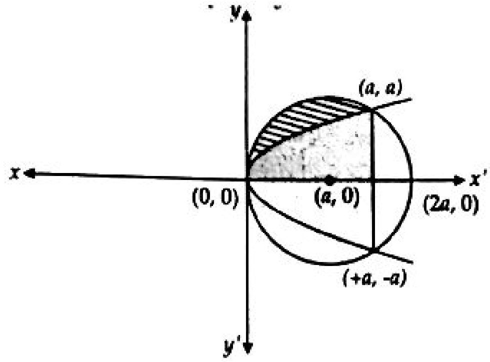
$x^2 + ax = 2ax$

Or $\quad x^2 - ax = 0$

Or $\quad x(x-a) = 0$

Or $\quad x = 0,\ x. = a$

Or $\quad y = 0,\ y = \pm a$



Point of intersection are (0, 0) (a, a), (a, -a)

$$x^2 + y^2 \le 2ax$$

(Area of shaded part)

Required Area = $\int_0^a y$ of circle $dx - \int_0^a y$ of parabola dx

$$= \int_0^a \sqrt{a^2 - (x-a)^2}\,dx - \int_0^a \sqrt{a}\sqrt{x}\,dx$$

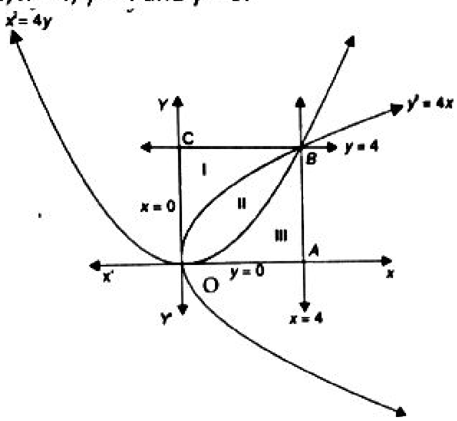
$$A = \left[\frac{x-a}{2}\sqrt{a^2-(x-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)\right]_0^a - \sqrt{a}\left[\frac{2x^{\frac{3}{2}}}{3}\right]_0^a$$

$$A = \left[0 - \left\{\frac{-a}{2}\sqrt{0} + \frac{a^2}{2}\sin^{-1}(-1)\right\}\right] - \frac{2}{3}\sqrt{a}\left[a^{\frac{3}{2}} - 0\right]$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2}\right) - \frac{2}{3}\sqrt{a} \times a\sqrt{a}$$

$$= \frac{\pi a^2}{4} - \frac{2a^2}{3} \text{ sq. units}$$

27. Soln. Let OABC be the square bounded by x = 0, x = 4, y = 4 and y = 0.



$$A(\text{OABC}) = 4 \times 4$$

= 16 sq. units

From, $y^2 = 4x$ and $x^2 = 4y$

$$\left(\frac{x^2}{4}\right)^2 = 4x$$

Or $\frac{x^4}{16} = 4x$

Or $x^4 - 64x = 0$

Or $x(x^3 - 64) = 0$

Or $x = 0$ or $x = 4$

When $x = 0,\ y = 0$

$x = 4,\ y = 4$

∴ Point of intersection of the two parabolas is (0, 0) and (4, 4).

Area of part III = $\int_0^4 y\,dx(\text{parabola } x^2 = 4y)$

$$= \int_0^4 \frac{x^2}{4}dx = \left[\frac{1}{4}\frac{x^3}{3}\right]_0^4$$

$$= \frac{1}{12}(64 - 0) = \frac{64}{12}$$

$$= \frac{16}{3} \text{ sq. units}$$

Area of I = Area of square – Area of II and III

$$= 16 - \int_0^4 \sqrt{4x}\,dx$$

$$= 16 - \frac{2 \times 2}{3}\left[x^{3/2}\right]_0^4$$

$$= 16 - \frac{32}{3} \text{ sq.units}$$

$$= \frac{16}{3} \text{ sq.units}$$

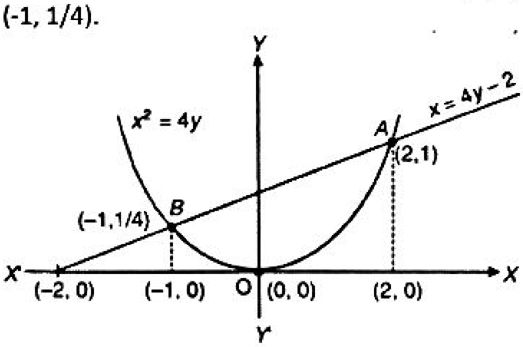
Area of II = Area of square – Area of I – Area of III

$$= 16 - \frac{16}{3} - \frac{16}{3} \text{ sq. units}$$

$$= \frac{16}{3} \text{ sq. units}$$

∴ The two curves divide the square into three equal parts.

28. Soln. ∴ Points of intersection are (2, 1) and (-1, 1/4).



Required Area of region (ΔOABO)

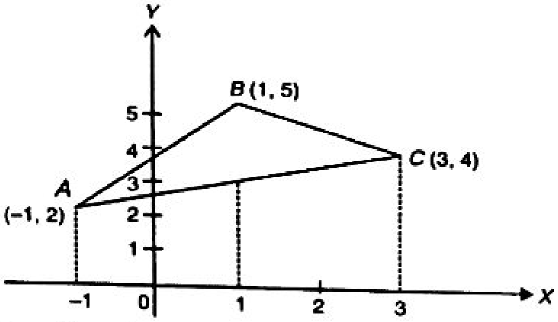
$$=\int_{-1}^{2}\frac{x+2}{4}dx-\int_{-1}^{2}\frac{x^2}{4}dx$$

$$=\frac{1}{4}\left[\frac{(x+2)^2}{2}-\frac{x^3}{3}\right]_{-1}^{2}$$

$$=\frac{1}{4}\left[\frac{16}{3}-\frac{5}{6}\right]=\frac{27}{24}\,sq.units$$

$$=\frac{9}{8}\,sq.units$$

29. Soln.



Equation of:

AB is: $y=\frac{1}{2}(3x+7)$

BC is: $y=\frac{1}{2}(11-x)$

AC is: $y=\frac{1}{2}(x+5)$

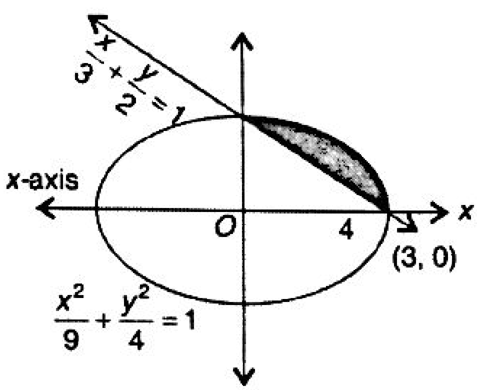
Required Area

$$=\frac{1}{2}\int_{-1}^{1}(3x+7)\,dx+\frac{1}{2}\int_{1}^{3}(11-x)dx-\frac{1}{2}\int_{-1}^{3}(x+5)\,dx$$

$$=\left[\frac{1}{12}(3x+7)^2\right]_{-1}^{1}-\frac{1}{4}\left[(11-x)^2\right]_{1}^{3}-\frac{1}{4}\left[(x+5)^2\right]_{-1}^{3}$$

$=7+9-12=4$ sq. units

30. Soln.



Area of shaded region

$$=\int_{0}^{3}\left\{\frac{2}{3}\sqrt{9-x^2}-\frac{2}{3}(3-x)\right\}dx$$

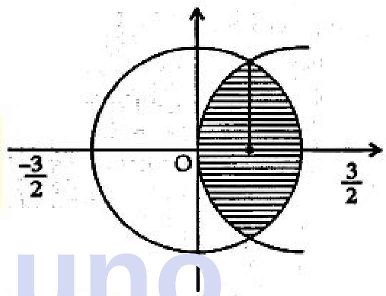
$$=\frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\frac{x}{3}+\frac{(3-x)^2}{2}\right]_{0}^{3}$$

$$=\frac{2}{3}\left[\left(0+\frac{9}{2}.\frac{\pi}{2}+0\right)-\left(0+0+\frac{9}{2}\right)\right]$$

$$=\frac{2}{3}\left(9\frac{\pi}{4}-\frac{9}{2}\right)$$

$$=3\left(\frac{\pi}{2}-1\right)\,sq.\ unit.$$

31. Soln. x-coordinate of point of intersection is, $x=\frac{1}{2}$



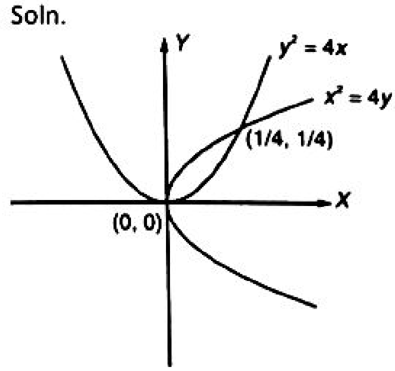
Required area

$$=2\left(\int_{0}^{\frac{1}{2}}2\sqrt{x}dx+\int_{\frac{1}{2}}^{3/2}\sqrt{\frac{9}{4}-x^2}\,dx\right)$$

$$=2\left[\frac{4}{3}x^{3/2}\Big|+\frac{x}{2}\sqrt{\frac{9}{4}-x^2}+\frac{9}{8}\sin^{-1}\frac{2x}{3}\Big|_{\frac{1}{2}}^{\frac{3}{2}}\right]$$

$$=\frac{\sqrt{2}}{6}+\frac{9}{4}\left(\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right)or\ \frac{\sqrt{2}}{6}+\frac{9}{4}\cos^{-1}\frac{1}{3}$$

32. Soln.



For intersection points, substitute

$$y^2=4x\ \ or\ \ y=2\sqrt{x}$$

$$x^2 = 4y$$

Or $x^2 = 4\times 2\sqrt{x}$ $\quad (\because y = 2\sqrt{x})$

Or $x^4 = 64x$

Or $x(x^3-64)=0$

Or $x = 0$

And $x = \pm 4$

When x = 0, y^2 = 4 x 0 or y = 0

When x = 4, y^2 = 4 x 4 or y = 4

∴ Points of intersection are (0, 0) and (4, 4).

Given, $y^2 = 4x \text{ or } y = 2\sqrt{x} = f(x)$

And $y = \frac{1}{4}x^2 = g(x)$

Where $f(x) \ge g(x)$ in (0, 4),

∴ Required Area = $\int_0^4 [f(x)-g(x)]dx$

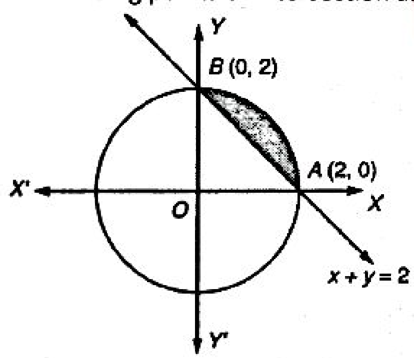
$$= \int_0^4 [2\sqrt{x} - \frac{1}{4}x^2]dx$$

$$= \left[\frac{4}{3}x^{3/2} - \frac{1}{12}x^3\right]_0^4$$

$$= \frac{4}{3}\times 4^{3/2} - \frac{1}{12}\times 4^3$$

$$= \frac{16}{3} \ sq.\ units$$

33. Soln. Finding points of intersection as x = 0, 2.



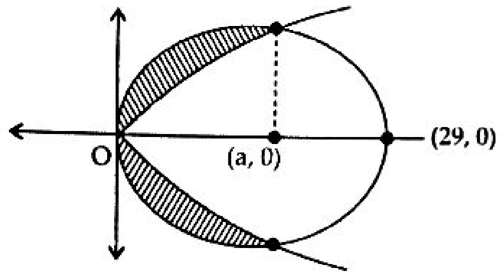
$$A = \int_0^2 \sqrt{4-x^2}dx - \int_0^2 (2-x)dx$$

$$= \left[\frac{x}{2}\times\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}\right]_0^2 - \left[2x - \frac{x^2}{2}\right]_0^2$$

$$= \left(0 + 2.\frac{\pi}{2}\right) - (4-2)$$

$$= (\pi - 2)\, sq.units$$

34. Soln.



x-coordinate of point of intersection is, x = a

Required area =

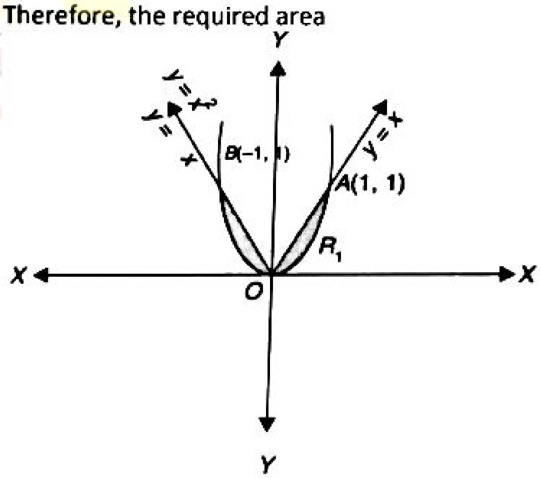
$$2\left[\int_0^a \left(\sqrt{a^2-(x-a)^2} - \sqrt{a}\sqrt{x}\right)dx\right]$$

$$= 2\left[\frac{x-a}{2}\sqrt{a^2-(x-a)^2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a} - 2\sqrt{a}\,\frac{x^{\frac{3}{2}}}{3}\right]_0^a$$

$$= 2[0 + 0\frac{-2}{3}\sqrt{a}\, a^{3/2} - \frac{a^2}{2}\sin^{-1}(-1)]$$

$$= \left(\frac{\pi}{2} - \frac{4}{3}\right)a^2$$

35. Soln. The required area is bounded between two curves y = x^2 and y = |x|. Both of these curves are symmetric about y-axis and shaded region in the fig. shows the region whose area is required. Therefore, the required area



A = 2 x Area of the region R_1

Now, to find the point of intersection of the curves $y = |x|$ and $y = x^2$, we solve them simultaneously.

Clearly, the region R_1 is in the first quadrant, where x > 0

∴ $|x| = x$ or $y = x$ ....(i)

and $y = x^2$ ....(ii)

Solving these two equations, we get

$$x = x^2$$

Or either x = 0 or x = 1

The limits are, when x = 0, y = 0 and when x = 1, y = 1

So, the point of intersection of the curves are O(0, 0) and A(1, 1).

Now, required Area = 2 x Area of line region R_1 =

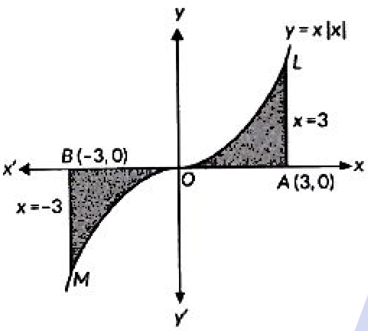
$$2\int_0^1[(\text{y of the line } y = x) - (\text{y of the parabola } y = x^2)]dx$$

$$= 2\int_0^1 (x - x^2)dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{3}\text{ Sq. units}$$

36. Soln. The equation of the curve is

$$y = x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

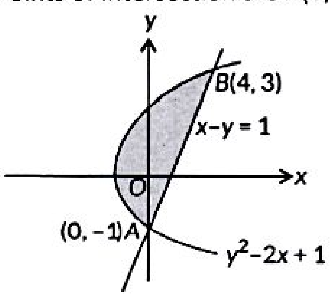


Required area = 2(Area of region shaded in first quadrant)

$$= 2\int_0^3 x^2 dx = 2\times\left[\frac{x^3}{3}\right]_0^3 = 2\times 9 = 18\,sq.units$$

Given $y^2 = 2x + 1$ and $x - y = 1$

Points of intersection are A(0, -1) and B(4, 3).

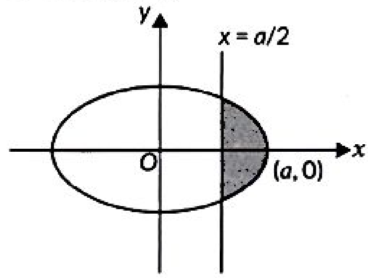


$$\therefore \text{ Required area} = \int_{-1}^{3}(1 + y)\,dy - \int_{-1}^{3}\left(\frac{y^2 - 1}{2}\right)dy$$

$$= \left[y + \frac{y^2}{2}\right]_{-1}^{3} - \left[\frac{1}{2}\left(\frac{y^3}{3} - y\right)\right]_{-1}^{3}$$

$$= \left[3 + \frac{9}{2} - \left(-1 + \frac{1}{2}\right)\right] - \frac{1}{2}\left[9 - 3 - \left(-\frac{1}{3} + 1\right)\right] = \frac{16}{3}\,sq.units$$

37. Soln. We have, an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, having centre at (0, 0)



Required area = area of shaded region

$$= 2\int_{a/2}^{a} y\,dx = \frac{2b}{a}\int_{\frac{a}{2}}^{a}\sqrt{a^2 - x^2}\,dx$$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$, Then, we get

Required area =

$$2ab\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta = ab\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(1 + \cos 2\theta)d\theta$$

$$= ab\left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = ab\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right]\text{ sq. units}$$

38. Soln. The tangent on

$x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is $x + \sqrt{3}y = 4$

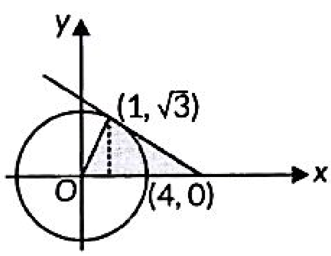
And equation of normal at $(1, \sqrt{3})$ is $y = x\sqrt{3}$

$$\text{Required area} = \int_0^1 x\sqrt{3}\,dx + \int_1^4 \frac{4 - x}{\sqrt{3}}dx$$

$$= \sqrt{3}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{\sqrt{3}}\left[4x - \frac{x^2}{2}\right]_1^4$$

$$= \sqrt{3}\times\frac{1}{2} + \frac{1}{\sqrt{3}}\left[4(4-1) - \frac{1}{2}(16-1)\right]$$

$$= \frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{2} = 2\sqrt{3}\,sq.units$$



First we find the equations of the sides of triangle ABC by using $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

The equation of AB is

$$y-5=\frac{7-5}{4-2}(x-2)\Rightarrow x-y+3=0 \quad ......(i)$$

The equation of BC is

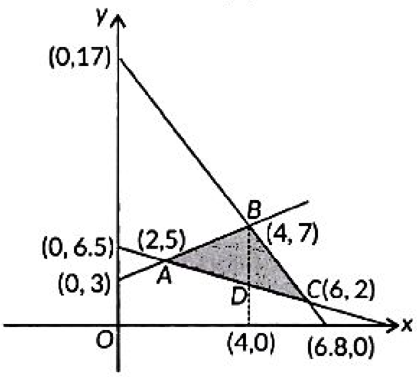
$$y-7=\frac{2-7}{6-4}(x-4)\Rightarrow 5x+2y-34=0$$

...........(ii)

The equation of side AC is

$$y-5=\frac{2-5}{6-2}(x-2)\Rightarrow 3x+4y-26=0$$

.............(iii)



Clearly, Area of

$$\Delta ABC = Area(\Delta ADB)+Area(\Delta BDC)$$

$$Area(\Delta ADB)=\int_2^4\left\{(x+3)-\left(\frac{26-3x}{4}\right)\right\}dx$$

Similarly, we have

$$Area(\Delta BDC)=\int_4^6\left\{\left(\frac{34-5x}{2}\right)-\left(\frac{26-3x}{4}\right)\right\}dx$$

$\therefore$ *Area of* ΔABC

$$=\int_2^4\left\{(x+3)-\left(\frac{26-3x}{4}\right)\right\}dx+\int_4^6\left\{\left(\frac{34-5x}{2}\right)-\left(\frac{26-3x}{4}\right)\right\}dx$$

$$=\frac{1}{4}\int_2^4(7x-14)\,dx+\frac{1}{4}\int_4^6(42-7x)dx$$

$$=\frac{1}{4}\left[\left[\frac{7x^2}{2}-14x\right]_2^4+\left[42x-\frac{7x^2}{2}\right]_4^6\right]$$

$$=\frac{1}{4}[\{(56-56)-(14-28)\}+\{(252-126)-(168-56)\}]$$

$$=\frac{1}{4}[14+126-112]=7\text{ sq.units}$$

SURE SHOT QUESTIONS

Chapter – 09 (Solution)

Differential Equations

➢ MCQ (1 mark)

1. Soln. (d): Since, the term $\sin\left(\frac{dy}{dx}\right)$ occurs in the given differential equation, therefore, its degree is not defined.

2. Soln. (d): The given differential equation can be written as $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=\left(\frac{d^2y}{dx^2}\right)^2$

Clearly, degree of given differential equation is 2.

3. Soln. (a): $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{\frac{1}{4}}+x^{\frac{1}{5}}=0$

Clearly, order of given differential equation is 2 and degree is not defined.

4. Soln. (b): We have, $y=A\cos\alpha x+B\sin\alpha x$ ....(i)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=-A\alpha\sin\alpha x+B\alpha\cos\alpha x=\alpha(-A\sin\alpha x+B\cos\alpha x)$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\alpha(-A\alpha\cos\alpha x-B\alpha\sin\alpha x)$$
$$=-\alpha^2(A\cos\alpha x+B\sin\alpha x)=-\alpha^2y \quad [\text{Using (i)}]$$

$$\Rightarrow \frac{d^2y}{dx^2}+\alpha^2y=0$$

5. Soln. (c): We have, $xdy-ydx=0\Rightarrow\frac{dy}{y}=\frac{dx}{x}$

On integration, we get

$$\log y=\log x+\log c\Rightarrow y=xc$$

Which is equation of a straight line passing through origin.

6. Soln. (c): $\cos x\frac{dy}{dx}+y\sin x=1$

$$\Rightarrow\frac{dy}{dx}+\frac{\sin x}{\cos x}.y=\frac{1}{\cos x}\Rightarrow\frac{dy}{dx}+\tan x.y=\sec x$$

$$\therefore I.F.=e^{\int\tan x dx}=e^{\log(\sec x)}=\sec x$$

7. Soln. (d): $\tan y\sec^2 xdx+\tan x\sec^2 ydy=0$

$$\Rightarrow\frac{\sec^2 xdx}{\tan x}=-\frac{\sec^2 yd}{\tan y}$$

On integration, we get

$$\log\tan x=-\log\tan y+\log k$$

$$\Rightarrow\log(\tan x.\tan y)=\log k\Rightarrow\tan x.\tan y=k$$

8. Soln. (a) : We have, $y=Ax+A^3$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=A$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=1$$

Clearly, degree of above differential equation is 1.

9. Soln. (c): $\frac{xdy}{dx} - y = x^4 - 3x$

$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = \frac{x^4 - 3x}{x} \Rightarrow \frac{dy}{dx} - \frac{1}{x}.y = x^3 - 3$

$\therefore\ I.F. = e^{-\int \frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$

10. Soln. (d): $\frac{dy}{dx} - y = 1 \Rightarrow \frac{dy}{dx} = 1 + y \Rightarrow \frac{dy}{1+y} = dx$

On integrating, we get

$\log(1+y) = x + c$

Now, $y(0) = 1 \Rightarrow \log 2 = c$

$\therefore\quad \log(1+y) = x + \log 2$

$\Rightarrow\quad \frac{1+y}{2} = e^x \Rightarrow 1 + y = 2e^x \Rightarrow y = 2e^x - 1$

11. Soln. (b): $\frac{dy}{dx} = \frac{y+1}{x-1} \Rightarrow \frac{dy}{y+1} = \frac{dx}{x-1}$

On integration, we get

$\log(y+1) + \log c = \log(x-1) \Rightarrow (y+1)c = (x-1)$

Now, $y(1) = 2 \Rightarrow 3c = 0 \Rightarrow c = 0$

$\therefore\quad x - 1 = 0 \Rightarrow x = 1$

Hence, only one solution exists.

12. Soln. (b): $y'y'' + y = \sin x$

$\Rightarrow\quad \left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2} + y = \sin x\right)$

Which is a second order differential equation.

13. Soln. (c):

$(1-x^2)\frac{dy}{dx} - xy = 1 \Rightarrow \frac{dy}{dx} - \frac{x}{1-x^2}.y = \frac{1}{1-x^2}$

$\therefore\ I.F. = e^{-\int \frac{x}{1-x^2}dx} = e^{\frac{1}{2}\int \frac{-2x}{1-x^2}dx}$

$= e^{\frac{1}{2}\log(1-x^2)} = e^{\log(1-x^2)^{\frac{1}{2}}} = \sqrt{1-x^2}$

14. Soln. (a): $e^x \cos y\ dx - e^x \sin y\ dy = 0$

$\Rightarrow e^x \cos y\ dx = e^x \sin y\, dy \Rightarrow dx = \tan y\, dy$

On integrating, we get

$x = \log(\sec y) + \log k$

$\Rightarrow\quad x = \log[(\sec y)k] \Rightarrow e^x = k \sec y$

$\Rightarrow\quad \frac{e^x}{\sec y} = k \Rightarrow e^x \cos y = k$

15. Soln. (a)

16. Soln. (b): $\frac{dy}{dx} + y = e^{-x}$

Clearly, it is a linear differential equation with

I.F. = $e^{\int dx} = e^x$

Now, solution is $y.e^x = \int e^x.e^{-x}dx + c$

$\Rightarrow ye^x = \int dx + c \Rightarrow ye^x = x + c \Rightarrow y = xe^{-x} + ce^{-x}$

$\therefore\quad y(0) = 0 \Rightarrow c = 0$

$\therefore\quad y = xe^{-x}$

17. Soln. (b): $\frac{dy}{dx} + y\tan x - \sec x = 0$

$\therefore\quad I.F. = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$

18. Soln. (b): $\frac{dy}{dx} = \frac{1+y^2}{1+x^2} \Rightarrow \frac{dy}{1+y^2} = \frac{dx}{1+x^2}$

On integrating, we get

$\tan^{-1} y = \tan^{-1} x + c \Rightarrow \tan^{-1} y - \tan^{-1} x = c$

$\Rightarrow \tan^{-1}\left(\frac{y-x}{1+xy}\right) = c \Rightarrow \frac{y-x}{1+xy} = \tan c = k(\text{say})$

$\Rightarrow\ y - x = k(1+xy)$

19. Soln. (b): $\frac{dy}{dx} + y = \frac{1+y}{x} \Rightarrow \frac{dy}{dx} + y = \frac{1}{x} + \frac{y}{x}$

$$\Rightarrow \frac{dy}{dx}+y-\frac{y}{x}=\frac{1}{x} \Rightarrow \frac{dy}{dx}+\left(1-\frac{1}{x}\right)y=\frac{1}{x}$$

$$\therefore\ I.F.=e^{\int\left(1-\frac{1}{x}\right)dx}=e^{x-\log x}=e^{x}e^{-\log x}=e^{x}e^{\log x^{-1}}$$

$$=e^{x}x^{-1}=\frac{e^{x}}{x}$$

20. Soln. (c): $y=ae^{mx}+be^{-mx}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=mae^{mx}-mbe^{-mx}=m(ae^{mx}-be^{-mx})$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=m(ame^{mx}+bme^{-mx})$$

$$=m^2(ae^{mx}+be^{-mx})=m^2y \quad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}-m^2y=0$$

21. Soln. (b): $\cos x \sin y\, dx+\sin x \cos y\, dy=0$

$$\frac{\cos x dx}{\sin x}=-\frac{\cos y dy}{\sin y}$$

On integrating, we get

$$\log \sin x=-\log \sin y+\log c$$

$$\Rightarrow \quad \log \sin x+\log \sin y=\log c \Rightarrow \sin x \sin y=c$$

22. Soln. (a): $x\frac{dy}{dx}+y=e^{x}$

$$\frac{dy}{dx}+\frac{1}{x}y=\frac{e^{x}}{x}$$

It is a linear differential equation with

$$I.F.=e^{\int\frac{1}{x}dx}=e^{\log x}=x$$

Now, solution is $y.x=\int\frac{e^{x}}{x}x\,dx+k$

$$\Rightarrow \ yx=e^{x}+k \Rightarrow \ y=\frac{e^{x}}{x}+\frac{k}{x}$$

23. Soln. (a): $x^2+y^2-2ay=0$ ....(i)

Differentiating w.r.t. x, we get

$$2x+2y\frac{dy}{dx}-2a\frac{dy}{dx}=0$$

$$\Rightarrow \ 2x+2y\frac{dy}{dx}-\left(\frac{x^2+y^2}{y}\right)\frac{dy}{dx}=0$$

$$\Rightarrow \left(\frac{x^2+y^2}{y}\right)\frac{dy}{dx}-2y\frac{dy}{dx}=2x$$

$$\Rightarrow \left(\frac{x^2+y^2-2y^2}{y}\right)\frac{dy}{dx}=2x$$

$$\Rightarrow \ (x^2-y^2)\frac{dy}{dx}=2xy$$

24. Soln. (b): $y=Ax+A^3$

Differentiating w.r.t. x, we get $\frac{dy}{dx}=A$

Again differentiating w.r.t. x, we get $\frac{d^2y}{dx^2}=0$

Which is differential equation of order 2.

25. Soln. (c): $\frac{dy}{dx}=2x.e^{x^2-y} \Rightarrow e^{y}dy=2xe^{x^2}dx$

On integrating we get $e^{y}=e^{x^2}+c$

26. Soln. (d): Slope of tangent, $\frac{dy}{dx}=\frac{x}{y} \Rightarrow ydy=xdx$

On integrating, we get

$$\frac{y^2}{2}=\frac{x^2}{2}+k \Rightarrow \ y^2-x^2=2k=c\text{(say)}$$

Which is equation of a rectangular hyperbola.

27. Soln. (d): $\frac{dy}{dx}+y=e^{-x}$

It is a linear differential equation with I.F. = $e^{\int dx}=e^{x}$

Now, solution is $y.e^x = \int e^x e^{-x} dx + c$

$\Rightarrow \; ye^x + x + c \Rightarrow y = (x+c)e^{-x}$

Now, $y(0) = 0 \Rightarrow c = 0$

Hence, $y = xe^{-x}$

28. Soln. (d)

29. Soln. (c): $\left[1 + \left(\frac{dy}{dx}\right)^2\right] = \frac{d^2y}{dx^2}$

Clearly, it is a second order differential equation with degree 1.

30. Soln. (a): $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

Using symbolic form of the equation

$(D^2 - 2D + 1)y = 0$, where $D = \frac{d}{d}$

$\Rightarrow D^2 - 2D + 1 = 0 \Rightarrow (D-1)^2 = 0$

$\Rightarrow D = 1, 1$

$\therefore$ Required solution is

$y = Axe^x + Be^x \Rightarrow y = (Ax+B)e^x$

31. Soln. (a): $\frac{dy}{dx} + y\tan x = \sec x$

It is a linear differential equation with

$I.F. = e^{\int \tan x dx} = e^{\log(\sec x)} = \sec x$

Now, solution is $y\sec x = \int \sec^2 x dx + c$

$\Rightarrow \; y\sec x = \tan x + c$

32. Soln. (a): $\frac{dy}{dx} + \frac{y}{x} = \sin x$

It is a linear differential equation with

I.F. $= e^{\int \frac{1}{x} dx} = e^{\log x} = x$

Now, solution is $y.x = \int x\sin x dx + c$

$\Rightarrow \; x.y = -x\cos x + \sin x + c$

$\Rightarrow \; x(y + \cos x) = \sin x + c$

33. Soln. (c) : $(e^x + 1)ydy = (y+1)e^x dx$

$\Rightarrow \frac{y}{y+1}dy = \frac{e^x}{e^x + 1}dx$

On integrating, we get

$$y - \log(y+1) = \log(e^x + 1) + \log k$$

$\Rightarrow \quad y = \log(y+1) + \log(e^x+1) + \log k$

$\Rightarrow \quad y = \log\{k(y+1)(e^x+1)\}$

34. Soln. (b): $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

$\Rightarrow \frac{dy}{dx} = e^{-y}(e^x + x^2) \Rightarrow e^y\, dy = (e^x + x^2)dx$

On integrating we get

$e^y = e^x + \frac{x^3}{3} + c \Rightarrow e^y - e^x = \frac{x^3}{3} + c$

35. Soln. (a): $\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{1}{(1+x^2)^2}$

It is a linear differential equation with

I.F. $= e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$

Now, solution is

$$y(1+x^2) = \int (1+x^2).\frac{1}{(1+x^2)^2}dx + c = \int \frac{dx}{1+x^2} + c$$

$\Rightarrow \; y(1+x^2) = \tan^{-1} x + c$

36. Soln. (b): [There is error in question, the given differential equation should be $\frac{d}{dx}\left(\frac{dy}{dx}\right)^3 = 0$]

The given differential equation is,

$$\frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^3\right)=0 \Rightarrow 3\left(\frac{dy}{dx}\right)^2\left(\frac{d^2y}{dx^2}\right)=0$$

∴ Order = 2 and degree = 1

So, required sum = 2 + 1 = 3

37. Soln. (b): We have, $\left(1+3\frac{dy}{dx}\right)^2=4\frac{d^3y}{dx^3}$

Here, order = 3 as highest order derivative is $\frac{d^3y}{dx^3}$

And degree = 1, as power of highest order derivative i.e.,

$\frac{d^3y}{dx^3}$ is 1.

38. Soln. (b) : Given that; $\frac{dy}{dx}=\frac{y+1}{x-1}\Rightarrow\frac{dy}{y+1}=\frac{dx}{x-1}$

On integrating both sides, we get

$$\int\frac{dy}{y+1}=\int\frac{dx}{x-1}$$

$$\Rightarrow \log(y+1)=\log(x-1)-\log C$$

$$\Rightarrow \log(y+1)+\log C=\log(x-1)$$

$$\Rightarrow C=\frac{x-1}{y+1}$$

Now, $y(1)=2\Rightarrow C=\frac{1-1}{2+1}=0$

∴ Required solution is x – 1 = 0

39. Soln. (a):In the particular solution of a differential equation of any order, there is no arbitrary constant because in the particular solution of any differential equation, we remove all the arbitrary constant by substituting some particular values.

40. Soln.

(d): We have, $x\frac{dy}{dx}-y=2x^2$

$$i.e., \frac{dy}{dx}-\frac{y}{x}=2x \quad \therefore I.F.=e^{\int\frac{-1}{x}dx}$$

$$=e^{-lnx}=e^{lnx^{-1}}=\frac{1}{x}$$

∴ Integrating factor is $\frac{1}{x}$.

41. Soln.

(c): We have, $(x+3y^2)\frac{dy}{dx}=y$

$$\Rightarrow \frac{x+3y^2}{y}=\frac{dx}{dy}\Rightarrow\frac{dx}{dy}-\frac{x}{y}=3y$$

This is a linear differential equation.

$$\therefore\ I.F.=e^{-\int\frac{dy}{y}}=e^{-\log y}=e^{\log y^{-1}}=\frac{1}{y}$$

42. Soln.

(c): The given differential equation is

$$\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)\right]^4$$

$$\Rightarrow 4\left(\frac{dy}{dx}\right)^3\frac{d^2y}{dx^2}=0$$

Here, m = 2 and n = 1

Hence, m + n = 3

➢ Assertion-Reasoning (1 mark)

43. Sol. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation: $\frac{dy}{dx}$ + py = Q

$\frac{dy}{dx}$ + y = x^2

On comparing with standard form

p = 1, Q = x^2

$\int pdx$

I.f. = e

= $e^{\int 1\cdot dx}$

= e^x

44. Sol. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation: Both A and R are true and R is the correct explanation of A.

45. Sol. **(b)** Both A and R are true but R is not the correct explanation of A.

Explanation: Squaring both sides of the given differential equation,

$$\left(\sqrt{\frac{d^2y}{dx^2}}\right)^2 = \left(\sqrt{\frac{dy}{dx}+5}\right)^2$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx} + 5$$

The highest order is 2 and its power is 1

∴ Order is 2, degree is 1

Hence, Assertion (A) is true.

The equation given in reason (R) is,

$$\left(\frac{1}{\frac{dy}{dx}}\right)^3 + 2\sqrt{y} = x$$

$$\Rightarrow \frac{1+2\sqrt{y}\left(\frac{dy}{dx}\right)^3}{\left(\frac{dy}{dx}\right)^3} = x$$

$$\Rightarrow 1 + 2\sqrt{y}\left(\frac{dy}{dx}\right)^3 = x\left(\frac{dy}{dx}\right)^3$$

Highest order is 1 and its power is 3

∴ Order is 1 and degree is 3.

Hence, reason (R) is also true.

46. Sol. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation: Both A and R are true and R is the correct explanation of A.

47. Sol. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation: $\frac{dy}{dx} - \frac{2}{x+1}y = (x+1)^3$

Hence

$$IF = e^{\int \frac{-2}{x+1}\cdot dx}$$

$$= e^{-2\ln(x+1)}$$

$$= e^{-\ln(x+1)^2}$$

$$= \frac{1}{(x+1)^2}$$

Thus the above differential equation changes to

$$\frac{1}{(x+1)^2}\cdot\frac{dy}{dx} - \frac{2}{(x+1)^3}\cdot y = (x+1)$$

$$\frac{d\left(\frac{y}{(x+1)^2}\right)}{dx} = (x+1)$$

$$\frac{y}{(x+1)^2} = \int (x+1)\cdot dx$$

$$\frac{y}{(x+1)^2} = \frac{x^2}{2} + x + c$$

$$\frac{2y}{(x+1)^2} = x^2 + 2x + c$$

$$\frac{2y}{(x+1)^2} = (x+1)^2 + C \text{ where } C = c - 1$$

➤ Case Study [4 Marks]

48. Soln.

(i) We have, $(x^2-y^2)dx+2xydy=0$

$$\Rightarrow \frac{dy}{dx} = -\frac{(x^2-y^2)}{2xy} = \frac{y^2-x^2}{2xy}$$

Now, putting $\frac{dy}{dx} = F(x,y)$ and find $F(\lambda x, \lambda y)$,

$$\Rightarrow F(x,y) = \frac{y^2-x^2}{2xy}$$

$$\therefore F(\lambda x, \lambda y) = \frac{\lambda^2y^2-\lambda^2x^2}{2\lambda^2xy} = \frac{y^2-x^2}{2xy} = F(x,y)$$

So, F(x, y) is homogenous function and the given differential equation is of the type $g\left(\frac{y}{x}\right)$

(ii) We have, $\frac{dy}{dx} = \frac{y^2-x^2}{2xy}$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore v + x\frac{dv}{dx} = \frac{v^2x^2-x^2}{2x.vx} = \frac{v^2-1}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2-1}{2v} - v = \frac{v^2-1-2v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = -\left(\frac{v^2+1}{2v}\right) \Rightarrow \int\frac{2v}{v^2+1}dv = -\int\frac{dx}{x}$$

$$\Rightarrow \int\frac{2v}{v^2+1}dv + \int\frac{dx}{x} = \log C$$

$$\Rightarrow \log|v^2+1| + \log x = \log C$$

$$\Rightarrow \log\left|\left(\frac{y^2+x^2}{x^2}\right)\times x\right| = \log C$$

$$\Rightarrow \frac{y^2+x^2}{x} = C \Rightarrow x^2+y^2 = Cx$$

Is the required general solution.

➢ Questions

1. Soln.

The given differential equation is

$$\left[\frac{d}{dx}(xy^2)\right].\frac{dy}{dx}+y=0$$

$$\Rightarrow \left[x.2y\frac{dy}{dx}+y^2.1\right]\frac{dy}{dx}+y=0$$

$$\Rightarrow 2x\left(\frac{dy}{dx}\right)^2+y^2\left(\frac{dy}{dx}\right)+y=0$$

$\therefore$ Its order is 1 and degree is 2.

$\therefore$ Required product = $1\times 2=2$.

2. Soln.

We have, $x\dfrac{dy}{dx}+(1+x\cot x)\,y=x$

$$\Rightarrow \frac{dy}{dx}+\frac{(1+x\cot x)}{x}y=1$$

Clearly it is a linear differential equation of the form,

$$\frac{dy}{dx}+Py=Q,\ where\ P=\frac{1+x\cot x}{x}\ and\ Q=1$$

$$\therefore\ I.F.'=e^{\int\left(\frac{1}{x}+\cot x\right)dx}$$

$$=e^{\log x+\log\sin x}=e^{\log(x\sin x)}=x\sin x.$$

3. Soln.

We have, $(1+x^2)\dfrac{dy}{dx}+2xy=\tan x$

$$\Rightarrow \frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{\tan x}{1+x^2}\qquad \text{.............(i)}$$

Clearly, it is a linear differential equation of the form

$$\frac{dy}{dx}+Py=Q,\ Where\ P=\frac{2x}{1+x^2}\ and\ Q=\frac{\tan x}{1+x^2}$$

$$\therefore\ I.F.=\int_e\frac{2x}{1+x^2}dx=e^{\log(1+x^2)}=1+x^2$$

$\therefore$ *Solution of* (i) is

$$y(1+x^2)=\int(1+x^2)\frac{\tan x}{(1+x^2)}dx+C$$

$$\Rightarrow\ y(1+x^2)=\int\tan x\,dx+C$$

$$\Rightarrow\ y(1+x^2)=\log|\sec x|+C$$

Also given, y(0) = 1

$\therefore\ 1=\log\sec 0+C\Rightarrow C=1$

Particular solution is

$$y(1+x^2)=\log|\sec x|+1$$

4. Soln.

We have, $x\cos\left(\dfrac{y}{x}\right)\dfrac{dy}{dx}=\left(\dfrac{y}{x}\right)\cos\left(\dfrac{y}{x}\right)+x;\ x\neq 0$

$$\Rightarrow \cos\left(\frac{y}{x}\right)\frac{dy}{dx}=\left(\frac{y}{x}\right)\cos\left(\frac{y}{x}\right)+1$$

.........(i)

This is a linear homogeneous differential equation

Put $y=vx\ \Rightarrow\ \dfrac{dy}{dx}=v.1+x.\dfrac{dv}{dx}$

Now (i) becomes

$$\cos v.\left[v+x\frac{dv}{dx}\right]=v\cos v+1$$

$$\Rightarrow\ x\cos v\frac{dv}{dx}=1\Rightarrow \cos v\,dv=\frac{dx}{x}$$

Integrating both sides, we get

$$\sin v=\log x+C\Rightarrow \sin\left(\frac{y}{x}\right)=\log x+C$$

Is the required solution.

5. Soln.

We have, $(x^3-3xy^2)dx=(y^3-3x^2y)dy$

$$\Rightarrow \frac{dy}{dx}=\frac{x^3-3xy^2}{y^3-3x^2y}\qquad \text{............(i)}$$

Put, $y=vx\Rightarrow \dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

$\therefore$ (i) becomes

$$v+x\frac{dv}{dx}=\frac{1-3v^2}{v^3-3v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-3v^2-v^4+3v^2}{v^3-3v}=\frac{1-v^4}{v(v^2-3)}$$

$$\Rightarrow \frac{v(v^2-3)\,dv}{1-v^4}=\frac{dx}{x}$$

$$\Rightarrow \int\frac{(v^3-3v)\,dv}{(1-v^2)(1+v^2)}=\int\frac{dx}{x} \quad \text{.......(ii)}$$

Now, let $\frac{v^3-3v}{(1-v^2)(1+v^2)}=\frac{Av+B}{1-v^2}+\frac{Cv+D}{1+v^2}$

.....(iii)

$$\Rightarrow v^3-3v=(Av+B)(1+v^2)+(Cv+D)(1-v^2)$$

Comparing coeff. Of like powers, we get

$$A-C=1, A+C=-3, B-D=0 \; and \; B+D=0$$

Solving these equations, we get A = -1, B = 0, C = -2, D = 0

From (ii) and (iii), we have

$$\int\frac{-v}{1-v^2}dv-\int\frac{2v}{1+v^2}dv=\int\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\log(1-v^2)-\log(1+v^2)=\log x+\log C_1$$

$$\Rightarrow \frac{\sqrt{1-v^2}}{1+v^2}=C_1x \Rightarrow x\frac{\left(\sqrt{x^2-y^2}\right)}{x^2+y^2}=C_1x$$

$$\Rightarrow x^2-y^2=C_1^2(x^2+y^2)^2$$

$$i.e., \; x^2-y^2=C(x^2+y^2)^2 \qquad (\text{where } C_1^2=C)$$

Which is the required solution.

6. Soln. Given differential equation is,

$$(1+x)^2\frac{dy}{dx}+y=\tan^{-1}x \quad \text{......(i)}$$

Divide by $(1+x^2)$ in above equation, we get

$$\frac{dy}{dx}+\frac{1}{1+x^2}y=\frac{\tan^{-1}x}{1+x^2}$$

The above equation is of the form $\frac{dy}{dx}+Py=Q$

Where, $P=\frac{1}{1+x^2}$ and $Q=\frac{\tan^{-1}x}{1+x^2}$

$$\therefore \quad I.F.=e^{\int Pdx}$$

$$=e^{\int\frac{1}{1+x^2}dx}$$

$$=e^{\tan^{-1}x}$$

Solution of the equation is given by,

$$y.I.F.=\int Q\,I.F.\,dx+c$$

$$ye^{\tan^{-1}x}=\int\frac{e^{\tan^{-1}x}\tan^{-1}x}{1+x^2}dx+c$$

..........(ii)

Put $\tan^{-1}x=t$

Differentiate both sides w.r.t. x

$$\frac{1}{1+x^2}=\frac{dt}{dx}$$

$$\Rightarrow \frac{dx}{1+x^2}=dt$$

$\therefore$ Equation (ii) becomes

$$ye^{\tan^{-1}x}=\int e^t.t\,dt+c$$

$$ye^{\tan^{-1}x}=te^t-\int 1.e^t.dt+c$$

$$=te^t-e^t+c$$

$$=\tan^{-1}x\,e^{\tan^{-1}x}-e^{\tan^{-1}x}+c$$

$$\Rightarrow ye^{\tan^{-1}x}=e^{\tan^{-1}x}(\tan^{-1}x-1)+c$$

$$\Rightarrow y=\tan^{-1}x-1+ce^{-\tan^{-1}x}$$

7. Soln. $(1+x^2)dy+2xydx=\cot x\,dx$

Dividing with $(1+x^2)dx$ on both sides, we get

$$\frac{dy}{dx}+\left(\frac{2x}{1+x^2}\right)y=\frac{\cot x}{1+x^2}$$

On comparing this equation with

$$\frac{dy}{dx}+Py=Q,$$

Where $P=\frac{2x}{1+x^2}, Q=\frac{\cot x}{1+x^2}$

Now, $\quad I.F.=e^{\int Pdx}=e^{\int\frac{2x}{1+x^2}dx}$

$$=e^{\log(1+x^2)}=1+x^2$$

Solution is

$$y(\text{I.F.})=\int Q.(\text{I.F.})dx+C$$

$$y(1+x^2)=\int\frac{\cot x}{(1+x^2)}.(1+x^2)\,dx+C$$

$$\Rightarrow \quad y(1+x^2)=\int\cot x+C$$

$$\Rightarrow \quad y(1+x^2)=\log|\sin x|+C$$

8. Soln. Given,

$$(x+1)\frac{dy}{dx}=2e^{-y}-1$$

$$\Rightarrow \quad \frac{dy}{2e^{-y}-1}=\frac{dx}{x+1}$$

$$\Rightarrow \quad \int\frac{e^{y}dy}{2-e^{y}}=\int\frac{dx}{x+1}$$

Put $2-e^{y}=t$

$$-e^{y}d^{y}=dt$$

$$e^{y}dy=-dt$$

$$\therefore \quad \int-\frac{dt}{t}=\int\frac{dx}{x+1}$$

$$\Rightarrow \quad -\log t=\log|x+1|+\log C$$

$$\Rightarrow \quad -\log t^{-1}=\log[C(x+1)]$$

$$\Rightarrow \quad \frac{1}{t}=C(x+1)$$

$$\Rightarrow \quad \frac{1}{2-e^{y}}=C(x+1) \qquad \text{..........(i)}$$

Put x = 0 and y = 0 in (i), we get

$$\Rightarrow \quad \frac{1}{2}=C$$

$$\therefore \quad \frac{1}{2-e^{y}}=\frac{1}{2}(x+1) \qquad \text{Ans.}$$

9. Soln.
(i) Order = 3, degree = 1
(ii) Order = 2, degree = 2

10. Sol.
Order is 4 but degree is not defined

11. Soln. Given $y\log y dx-xdy=0$

$$\Rightarrow \quad y\log y dx=xdy$$

$$\Rightarrow \quad \frac{dx}{x}=\frac{dy}{y\log y}$$

$$\Rightarrow \quad \int\frac{dy}{y\log y}=\int\frac{dx}{x}$$

Let $\log y=t \quad \Rightarrow \quad \frac{1}{y}dy=dt$

$$\Rightarrow \quad \int\frac{dt}{t}=\log x+C_1$$

$$\Rightarrow \quad \log t=\log x+\log C \qquad [\text{Where } C_1=\log C]$$

$$\Rightarrow \quad t=Cx$$

$$\Rightarrow \quad \log y=Cx$$

$$\Rightarrow \quad y=e^{Cx}$$

12. Soln. Given differential equation,

$$e^{x}\tan y\,dx+(1-e^{x})\sec^{2}y\,dy=0$$

$$\Rightarrow \quad e^{x}\tan y\,dx=-(1-e^{x})\sec^{2}y\,dy$$

$$\therefore \quad dy=\frac{e^{x}}{e^{x}-1}\cdot\frac{\tan y}{\sec^{2}y}dx \Rightarrow \frac{\sec^{2}y\,dy}{\tan y}=\frac{e^{x}}{e^{x}-1}dx$$

Integrating both sides, we get

$$\int\frac{\sec^{2}y\,dy}{\tan y}=\int\frac{e^{x}}{e^{x}-1}dx$$

$$\Rightarrow \quad \log|\tan y|=\log|e^{x}-1|+\log C$$

$$\therefore \quad \log|\tan y|=\log|(e^{x}-1)C|$$

$$\therefore \quad \tan y=(e^{x}-1)C$$

13. Soln. Given $\frac{dy}{dx}=1+x^{2}+y^{2}+x^{2}y^{2}$

$$\Rightarrow \quad \frac{dy}{dx}=(1+x^{2})+y^{2}(1+x^{2})$$

$$\Rightarrow \quad \frac{dy}{dx}=(1+x^{2})(1+y^{2})$$

$$\Rightarrow \quad (1+x^{2})dx=\frac{dy}{(1+y^{2})}$$

Integrating both sides, we get

$$\int(1+x^{2})dx=\int\frac{dy}{(1+y^{2})}$$

$$\Rightarrow \quad \int dx+\int x^{2}dx=\int\frac{dy}{(1+y^{2})}$$

$$\Rightarrow \quad x+\frac{x^{3}}{3}+C=\tan^{-1}y$$

Putting y = 1 and x = 0, we get

$$\Rightarrow \quad \tan^{-1}(1)=0+0+C$$

$$\Rightarrow \quad C=\tan^{-1}(1)=\frac{\pi}{4}$$

Therefore, required particular solution is

$$\tan^{-1}=x+\frac{x^{3}}{3}+\frac{\pi}{4}$$

14. Soln. We have,

$$(1+e^{x/y})dx+e^{x/y}\left(1-\frac{x}{y}\right)dy=0$$

$$\Rightarrow \quad \left(1+e^{x/y}\right)dx = -e^{x/y}\left(1-\frac{x}{y}\right)dy$$

$$\therefore \quad \frac{dx}{dy} = \frac{-e^{x/y}\left(1-\frac{x}{y}\right)}{1+e^{x/y}} = g\left(\frac{x}{y}\right) \quad ......(i)$$

Here, RHS of differential equation is of the form $g\left(\frac{x}{y}\right)$, so it is a homogeneous function of degree zero.

Now, we put x = vy and, $\frac{dx}{dy} = v + y\frac{dv}{dy}$

From (i), we get $\quad v + y\frac{dv}{dy} = \frac{-e^{v}(1-v)}{1+e^{v}}$

$$y\frac{dv}{dy} = \frac{-e^{v}(1-v)}{1+e^{v}} - v = \frac{-(v+e^{v})}{1+e^{v}}$$

$$\Rightarrow \quad \frac{1+e^{v}}{-(v+e^{v})}dv = \frac{dy}{y}$$

On integrating both sides, we get

$$-\log|v+e^{v}| + \log C = \log|y| \Rightarrow \log C = \log|y| + \log|v+e^{v}|$$

$$\log C = \log|y(v+e^{v})| = \log\left|y\left(\frac{x}{y}+e^{x/y}\right)\right|$$

$$\Rightarrow \quad C = y\left(\frac{x}{y}+e^{x/y}\right) \quad or \quad C = x + ye^{x/y}$$

Hence, $x + ye^{x/y} = C$ is the required solution.

15. Soln. The given differential equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

.........(i)

Now, (i) is a linear differential equation of the form $\frac{dy}{dx} + Px = Q$,

Where,

$$P = \frac{1}{1+y^2} \; and \; Q = \frac{\tan^{-1}y}{1+y^2}$$

Therefore, $\quad IF = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

Thus, the solution of the given differential equation is

$$xe^{\tan^{-1}y} = \int\left(\frac{\tan^{-1}y}{1+y^2}\right)e^{\tan^{-1}y}dy + C$$

.........(ii)

Let $I = \int\left(\frac{\tan^{-1}y}{1+y^2}\right)e^{\tan^{-1}y}dy$

Substituting $\tan^{-1}y = t$ so that $\left(\frac{1}{1+y^2}\right)dy = dt$, we get

$$I = \int t\,e^{t}dt = t\,e^{t} - \int 1.e^{t}dt = t\,e^{t} - e^{t} \equiv e^{t}(t-1)$$

Or $\quad I = e^{\tan^{-1}y}(\tan^{-1}y - 1)$

Substituting the value of I in equation (ii), we get

$$x.e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y-1) + C$$

Or $\quad x = (\tan^{-1}y - 1) + Ce^{-\tan^{-1}y}$

Which is the general solution of the given differential equation.

16. Soln. The given differential equation is

$$(x^2-1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2x}{x^2-1}y = \frac{2}{(x^2-1)^2}$$

.......(i)

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{2x}{x^2-1}$ and $Q = \frac{2}{(x^2-1)^2}$

$$\therefore \quad IF = e^{\int P dx} = e^{\int 2x/(x^2-1)dx} = e^{\log(x^2-1)} = (x^2-1)$$

Multiplying both sides of (i) by $IF = (x^2-1)$, we get $(x^2-1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}$

Integrating both sides, we get

$$y(x^2-1) = \int \frac{2}{x^2-1}dx + C$$

$$\left[U\sin g : y(\text{IF}) = \int Q.(\text{IF})\,dx + C\right]$$

$$\Rightarrow \quad y(x^2-1)=\frac{2}{2}\log\left|\frac{x-1}{x+1}\right|+C$$

$$\Rightarrow \quad y(x^2-1)=\log\left|\frac{x-1}{x+1}\right|+C$$

This is the required solution.

17. Soln. We have, $\left(1+x^2\right)\frac{dy}{dx}+y=\tan^{-1}x$

Dividing each term by $\left(1+x^2\right)$, we get

$$\frac{dy}{dx}+\frac{1}{1+x^2}.y=\frac{\tan^{-1}x}{1+x^2}$$

Clearly, it is linear equation of the form

$\frac{dy}{dx}+P.y=Q$, where $P=\frac{1}{1+x^2}$ and

$Q=\frac{\tan^{-1}x}{1+x^2}$

$\therefore$ Integrating factor,

$IF=e^{\int P\,dx}=e^{\int\frac{1}{1+x^2}dx}=e^{\tan^{-1}x}$

Therefore, solution of given differential equation is $y\times I.F.=\int Q\times I.F.\,dx$

$$\Rightarrow \quad y.e^{\tan^{-1}x}=\int\frac{\tan^{-1}x}{1+x^2}.e^{\tan^{-1}x}dx$$

$$Put \quad I=\int\frac{\tan^{-1}x\,e^{\tan^{-1}x}}{1+x^2}dx$$

$$Put \quad e^{\tan^{-1}x}=t \quad \Rightarrow \quad \frac{e^{\tan^{-1}x}}{1+x^2}dx=dt$$

Also $\tan^{-1}x=\log t$

$$\Rightarrow \quad I=\int\log t\,dt=t\log t-t+C$$

[Integrating by parts]

$$\Rightarrow \quad I=e^{\tan^{-1}x}.\tan^{-1}x-e^{\tan^{-1}x}+C$$

Hence, required solution is

$$y.e^{\tan^{-1}x}=e^{\tan^{-1}x}(\tan^{-1}x-1)+C$$

$$\Rightarrow \quad y=(\tan^{-1}x-1)+Ce^{-\tan^{-1}x}$$

18. Soln. Given differential equation is

$$x\frac{dy}{dx}\sin\frac{y}{x}+x-y\sin\frac{y}{x}=0$$

Dividing both sides by $x\sin\frac{y}{x}$, we get

$$\frac{dy}{dx}+\cos ec\frac{y}{x}-\frac{y}{x}=0 \quad\Rightarrow\quad \frac{dy}{dx}=\frac{y}{x}-\cos ec\frac{y}{x}$$

Let $F(x,y)=\frac{y}{x}-\cos ec\frac{y}{x}$

$$\therefore \quad F(\lambda x,\lambda y)=\frac{\lambda y}{\lambda x}-\cos ec\frac{\lambda y}{\lambda x}=\lambda^0\left[\frac{y}{x}-\cos ec\frac{y}{x}\right]=\lambda^0F(x,y)$$

Hence, differential equation (i) is homogenous.

Let

$$y=vx \;\Rightarrow\; \frac{y}{x}=v \;\Rightarrow\; \frac{dy}{dx}=v+x.\frac{dv}{dx}$$

Now, equation (i) becomes

$$v+x.\frac{dv}{dx}=\frac{vx}{x}-\cos ec\frac{vx}{x}$$

$$v+x.\frac{dv}{dx}=v-\cos ec v\Rightarrow \quad x.\frac{dv}{dx}=-\cos ec v$$

$$\Rightarrow \quad -\sin v\,dv=\frac{dx}{x} \;\Rightarrow\; -\int\sin v\,dv=\int\frac{dx}{x}$$

$$\Rightarrow \quad \cos v=\log|x|+C\Rightarrow \cos\frac{y}{x}=\log|x|+C$$

Putting $y=\frac{\pi}{2}$, $x=1$ in (ii), we get

$$\therefore \quad \cos\frac{\pi}{2}=\log 1+C \quad\Rightarrow\quad 0=0+C \quad\Rightarrow\quad C=0$$

Hence, particular solution is

$$\cos\frac{y}{x}=\log|x|+0 \quad \text{i.e.,} \quad \cos\frac{y}{x}=\log|x|$$

19. Soln. Given differential equation is,

$$\cos^2x.\frac{dy}{dx}+y=\tan x \;\Rightarrow\; \frac{dy}{dx}+y\sec^2x=\tan x.\sec^2x$$

Given differential equation is a linear differential equation of the type $\frac{dy}{dx}+Py=Q$ where

$P=\sec^2x,\ Q=\tan x.\sec^2x$

$$IF=e^{\int P\,dx}=e^{\int\sec^2x\,dx}=e^{\tan x}$$

$\therefore$ Solution is given by

$$e^{\tan x}y=\int\tan x.\sec^2x.e^{\tan x}dx$$

Let $I=\int\tan x.\sec^2x.e^{\tan x}dx$

Put $\tan x=t,\ \sec^2x\,dx=dt$, we get

$$I=\int t\,e^t\,dt$$

$$\therefore \quad =te^t-\int e^t dt=t\,e^t-e^t+C$$

[Integrating by parts]

$$=\tan x\,e^{\tan x}-e^{\tan x}+C$$

Hence,

$$e^{\tan x} y = e^{\tan x}(\tan x - 1) + C \quad \Rightarrow \quad y = \tan x - 1 + Ce^{-\tan x}$$

20. Soln. Given

$$\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$$

By simplifying the equation, we get

$$xy\frac{dy}{dx} = -\sqrt{1+x^2+y^2+x^2y^2}$$

$$\Rightarrow \quad xy\frac{dy}{dx} = -\sqrt{(1+x^2)(1+y^2)} = -\sqrt{(1+x^2)}\sqrt{(1+y^2)}$$

$$\Rightarrow \quad \frac{y}{\sqrt{1+y^2}}dy = -\frac{\sqrt{1+x^2}}{x}dx$$

Integrating both sides, we get

$$\int \frac{y}{\sqrt{1+y^2}}dy = -\int \frac{\sqrt{1+x^2}}{x}dx$$

..........(i)

Let $1+y^2 = t \quad \Rightarrow \quad 2y\,dy = dt,$

(For LHS)

And

$1+x^2 = m^2 \quad \Rightarrow \quad 2xdx = 2m\,dm \Rightarrow x\,dx = m\,dm$

(For RHS)

$$\therefore (i) \Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{t}}dt = -\int \frac{m}{m^2-1}.m\,dm$$

$$\Rightarrow \quad \frac{1}{2}\frac{t^{1/2}}{1/2} + \int \frac{m^2}{m^2-1}dm = 0$$

$$\Rightarrow \quad \sqrt{t} + \int \frac{m^2+1-1}{m^2-1}dm = 0$$

$$\Rightarrow \quad \sqrt{t} + \int\left(1+\frac{1}{m^2-1}\right)dm = 0$$

$$\Rightarrow \quad \sqrt{t} + m + \frac{1}{2}\log\left|\frac{m-1}{m+1}\right| = 0$$

Now, substituting these value of t and m, we get

$$\sqrt{1+y^2} + \sqrt{1+x^2} + \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + C = 0$$

21. Soln. We have

$$ydx + x\log\left(\frac{y}{x}\right)dy - 2x\,dy = 0$$

Simplifying the above equation, we get

$$\left[x\log\left(\frac{y}{x}\right) - 2x\right]dy = -y\,dx \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)} \quad ....(i)$$

Let $F(x,y) = \dfrac{y}{2x - x\log\left(\frac{y}{x}\right)}$

$$F(\mu x, \mu y) = \frac{\mu y}{2\mu x + \mu x\log\left(\frac{\mu y}{\mu x}\right)} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)} = \mu^0 F(x,y)$$

$\therefore$ Function F(x, y) is homogeneous and hence the equation is homogeneous.

Let $y = vx \Rightarrow \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Substituting in equation (i), we get

$$v + x\frac{dv}{dx} = \frac{vx}{2x - x\log v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{2-\log v} - v \quad \Rightarrow \quad x\frac{dv}{dx} = \frac{v\log v - v}{2-\log v}$$

$$\Rightarrow \quad \frac{2-\log v}{v\log v - v}dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{2-\log v}{v\log v - v}dv = \int \frac{dx}{x} \quad \Rightarrow \quad \int \frac{1+(1+\log v)}{v(\log v - 1)}dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{dv}{v(\log v - 1)} - \int \frac{dv}{v} = \int \frac{dx}{x}$$

Let $\log v - 1 = m \quad \Rightarrow \quad \dfrac{1}{v}dv = dm$

$$\Rightarrow \quad \int \frac{1}{m}dm - \int \frac{1}{v}dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \log|m| - \log|v| = \log|x| + \log|C|$$

$$\Rightarrow \quad \log\left|\frac{m}{v}\right| = \log|Cx| \Rightarrow \frac{m}{v} = Cx \Rightarrow (\log v - 1) = vCx$$

$$\Rightarrow \quad \left[\log\left(\frac{y}{x}\right) - 1\right] = Cy$$

Which is the required solution.

22. Soln. Given, $(x-y)\dfrac{dy}{dx} = x + 2y$

By simplifying the above equation, we get

$$\frac{dy}{dx}=\frac{x+2y}{x-y}$$

.............(i)

Let $F(x,y)=\frac{x+2y}{x-y}$

Then

$$F(\lambda x,\lambda y)=\frac{\lambda x+2\lambda y}{\lambda x-\lambda y}=\frac{\lambda(x+2y)}{\lambda(x-y)}=\lambda^0 F(x,y)$$

F(x, y) is homogeneous function and hence given differential equation is homogeneous.

Now, Let $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

Substituting these values in equation (i), we get

$$v+x\frac{dv}{dx}=\frac{x+2vx}{x-vx}$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{1+2v}{1-v}-v=\frac{1+2v-v+v^2}{1-v}=\frac{1+v+v^2}{1-v}$$

$$\Rightarrow \quad \frac{1-v}{1+v+v^2}dv=\frac{dx}{x}$$

By integrating both sides, we get

$$\int\frac{1-v}{v^2+v+1}dv=\int\frac{dx}{x}$$

........(ii)

LHS $\int\frac{1-v}{v^2+v+1}dv$

Let $1-v=A(2v+1)+B=2Av+(A+B)$

Comparing coefficients of both sides, we get

$$2A=-1, \qquad A+B=1$$

Or $A=-\frac{1}{2},\ B=\frac{3}{2}$

$$\therefore \int\frac{1-v}{v^2+v+1}dv=\int\frac{-\frac{1}{2}(2v+1)+\frac{3}{2}}{v^2+v+1}dv$$

$$=-\frac{1}{2}\int\frac{2v+1}{v^2+v+1}dv+\frac{3}{2}\int\frac{dv}{v^2+v+1}$$

$$=-\frac{1}{2}\int\frac{(2v+1)}{v^2+v+1}dv+\frac{3}{2}\int\frac{dv}{\left(v+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$=-\frac{1}{2}\log|v^2+v+1|+\frac{3}{2}\times\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{v+\frac{1}{2}}{\sqrt{3}/2}\right)$$

Now, substituting it in equation (ii), we get

$$-\frac{1}{2}\log|v^2+v+1|+\sqrt{3}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right)=\log x+C$$

$$\Rightarrow \quad -\frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{y}{x}+1\right|+\sqrt{3}\tan^{-1}\left(\frac{\frac{2y}{x}+1}{\sqrt{3}}\right)=\log x+C$$

$$\Rightarrow \quad -\frac{1}{2}\log|x^2+xy+y^2|+\frac{1}{2}\log x^2+\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)=\log x+C$$

$$\Rightarrow \quad -\frac{1}{2}\log|x^2+xy+y^2|+\sqrt{3}\tan^{-1}\left(\frac{2y-x}{\sqrt{3}x}\right)=C$$

23. Soln. We have

$$\left(\cot^{-1}y+x\right)dy=\left(1+y^2\right)dx$$

This can be written as

$$\frac{dx}{dy}=\frac{\cot^{-1}y+x}{1+y^2}=\frac{\cot^{-1}y}{1+y^2}+\frac{x}{1+y^2}$$

$$\Rightarrow \quad \frac{dy}{dx}-\frac{1}{1+y^2}.x=\frac{\cot^{-1}y}{1+y^2}$$

It is linear differential equation of the form

$\frac{dx}{dt}+Px=Q$, where

$$P=-\frac{1}{1+y^2} \ and \ Q=\frac{\cot^{-1}y}{1+y^2}$$

$$\therefore \quad I.F.=e^{\int\frac{1}{1+y^2}dy}=e^{\cot^{-1}y}$$

Therefore, required solution of differential equation is

$$x.e^{\cot^{-1}y}=\int\frac{\cot^{-1}y}{1+y^2}.e^{\cot^{-1}y}dy+C$$

$$\Rightarrow \quad x.e^{\cot^{-1}y}=I+C \qquad ....(i)$$

Where, $I=\int\frac{\cot^{-1}y}{1+y^2}.e^{\cot^{-1}y}dy$

Let $\cot^{-1}y=t$

$$-\frac{1}{1+y^2}dt=dt \qquad \Rightarrow \qquad \frac{1}{1+y^2}dy=-dt$$

$$\Rightarrow \quad I=-\int t.e^t dt=-\left[t.e^t-\int e^t dt\right]=-t.e^t+e^t$$

$$=e^t(1-t)=e^{\cot^{-1}y}\left(1-\cot^{-1}y\right)$$

Hence, required solution is

$$x.e^{\tan^{-1}y} = e^{\cot^{-1}y}(1-\cot^{-1}y)+C$$
$$x = (1-\cot^{-1}y)+Ce^{-\cot^{-1}y}$$

24. Soln. The given differential equation is

$$\left(1+x^3\right)\frac{dy}{dx}+6x^2y=(1+x^2) \Rightarrow \frac{dy}{dx}+\frac{6x^2}{\left(1+x^3\right)}y=\frac{1+x^2}{1+x^3}$$

It is in the form of $\frac{dy}{dx}+Py=Q$, where

$$P=\frac{6x^2}{\left(1+x^3\right)}, Q=\frac{1+x^2}{1+x^3}$$

$$\therefore \quad IF = e^{\int P dx} = e^{\int \frac{6x^2}{1+x^3}dx}$$
$$= e^{2\int\frac{3x^2}{1+x^3}dx} = e^{2\int\frac{dt}{t}} \qquad [\text{Let } 1+x^3=t \Rightarrow 3x^2dx=dt]$$
$$= e^{2\log t} = e^{\log t^2} = t^2$$
$$= (1+x^3)^2$$

Therefore, general solution is

$$y.(1+x^3)^2 = \int\frac{1+x^2}{1+x^3}\times(1+x^3)^2dx+C$$
$$= \int\left(1+x^2\right)\left(1+x^3\right)dx+C = \int\left(x^5+x^3+x^2+1\right)dx+C$$
$$y.(1+x^3)^2 = \frac{x^6}{6}+\frac{x^4}{4}+\frac{x^3}{3}+x+C$$

Putting y = 1, x = 1, we get

$$\therefore 4 = \frac{1}{6}+\frac{1}{4}+\frac{1}{3}+1+C \Rightarrow C = 4-\frac{1}{6}-\frac{1}{4}-\frac{1}{3}-1=\frac{9}{4}$$

$\therefore$ Required particular solution is

$$y(1+x^3)^2 = \frac{x^6}{6}+\frac{x^4}{4}+\frac{x^3}{3}+x+\frac{9}{4}.$$

25. Soln. Given:

$$2y.e^{x/y}dx+(y-2xe^{x/y})dy=0$$

$$\Rightarrow \quad \frac{dx}{dy} = -\frac{y-2xe^{x/y}}{2y.e^{x/y}} \Rightarrow \frac{dx}{dy}=\frac{2x\,e^{x/y}-y}{2y.e^{x/y}}$$

$$Let \;\; F(x,y)=\frac{2x.e^{x/y}-y}{2y.e^{x/y}}$$

$$\therefore \;\; F(\lambda x,\lambda y)=\frac{2\lambda x.e^{\lambda x/\lambda y}-\lambda y}{2\lambda y.e^{\lambda x/\lambda y}}=\lambda^0\frac{2x\,e^{x/y}-y}{2y\,e^{x/y}}=\lambda^0.F(x,y)$$

Hence, given differential equation is homogeneous.

Now, $\quad \frac{dx}{dy}=\frac{2x\,e^{x/y}-y}{2y.e^{x/y}}$

............(i)

Let $x=vy \;\; \Rightarrow \;\; \frac{dx}{dy}=v+y\frac{dv}{dy}$

$$\therefore (i) \;\; \Rightarrow \;\; v+y.\frac{dv}{dy}=\frac{2vy.e^{\frac{vy}{y}}-y}{2y.e^{\frac{vy}{y}}}$$

$$\Rightarrow \;\; y.\frac{dv}{dy}=\frac{y(2\,v\,e^v-1)}{2y.e^v}-v \Rightarrow y.\frac{dv}{dy}=\frac{2v.e^v-1}{2e^v}-v$$

$$\Rightarrow \;\; y\frac{dv}{dy}=-\frac{1}{2e^v} \;\; \Rightarrow \;\; 2y\,e^v dv=-dy$$

$$\Rightarrow \;\; 2\int e^v dv=-\int\frac{dy}{y} \;\; \Rightarrow \;\; 2e^v=-\log y+C$$

$$\Rightarrow \;\; 2e^{\frac{y}{x}}+\log y=C$$

When $\quad$ x = 0, y = 1

$$\therefore \quad 2e^0+\log 1=C \; or \; C=2$$

Hence, the required solution is

$$2e^{x/y}+\log y=2 \qquad \Rightarrow \qquad \log C=2$$

26. Soln. Given differential equation is

$$\frac{dy}{dx}=\frac{x(2\log x+1)}{\sin y+y\cos y}$$

$$\Rightarrow \quad (\sin y+y\cos y)dy=x(2\log x+1)dx$$
$$\Rightarrow \quad \int\sin y\,dy+\int y\cos y\,dy=2\int x\log x\,dx+\int x\,dx$$
$$\Rightarrow \quad \int\sin y\,dy+[y\sin y-\int\sin y\,dy]=2\left[\log x\frac{x^2}{2}-\int\frac{1}{x}.\frac{x^2}{2}dx\right]+\int x\,dx$$
$$\Rightarrow \quad \int\sin y\,dy+y\sin y-\int\sin y\,dy=x^2\log x-\int x\,dx+\int x\,dx+C$$
$$\Rightarrow \quad y\sin y=x^2\log x+C \qquad ....(i)$$

It is general solution.

For particular solution, we put $y=\frac{\pi}{2}$ when x = 1

(i) Becomes $\frac{\pi}{2}\sin\frac{\pi}{2}=1.\log 1+C$

$$\frac{\pi}{2}=C \qquad [\because \log 1=0]$$

Putting the value of C in (i), we get the required particular solution

$$y\sin y=x^2\log x+\frac{\pi}{2}.$$

27. Soln. We have,

$$x\cos\left(\frac{y}{x}\right)\frac{dy}{dx}=y\cos\left(\frac{y}{x}\right)+x; x\neq 0$$

$$\Rightarrow \quad \cos\left(\frac{y}{x}\right).\frac{dy}{dx}=\left(\frac{y}{x}\right)\cos\left(\frac{y}{x}\right)+1$$

..........(i)

This is a linear homogeneous differential equation

Put $y = vx \Rightarrow \frac{dy}{dx} = v.1 + x.\frac{dv}{dx}$

Now (i) becomes

$$\cos v.\left[v + x\frac{dv}{dx}\right] = v\cos v + 1$$

$$\Rightarrow \quad x\cos v\frac{dv}{dx} = 1 \Rightarrow \cos v dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\sin v; = \log x + C \Rightarrow \sin\left(\frac{y}{x}\right) = \log x + C$$

Is the required solution.

28. Soln. We have,

$$\left(\tan^{-1} y - x\right)dy = \left(1 + y^2\right)dx$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\tan^{-1} y - x}{1 + y^2} \Rightarrow \frac{dx}{dy} + \frac{1}{1 + y^2}.x = \frac{\tan^{-1} y}{1 + y^2}$$

This is a linear differential equation of the form

$$\frac{dx}{dy} + Px = Q, \text{ where } P = \frac{1}{1 + y^2} \text{ and } Q = \frac{\tan^{-1} y}{1 + y^2}$$

$$I.F. = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

$\therefore$ Required solution is,

$$x.e^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}.\tan^{-1} y}{1 + y^2} dy + C$$

Put $\tan^{-1} y = t \Rightarrow \left(\frac{1}{1 + y^2}\right)dy = dt$

$\therefore$ (i) becomes,

$$x.e^{\tan^{-1} y} = \int e^t.t\,dt + C$$

$$\Rightarrow \quad x.e^{\tan^{-1} y} = t.e^t - \int 1.e^t dt + C$$

$$\Rightarrow \quad x.e^{\tan^{-1} y} = t.e^t - e^t + C$$

$$\Rightarrow \quad x.e^{\tan^{-1} y} = \tan^{-1} y.e^{\tan^{-1} y} - e^{\tan^{-1} y} + C$$

$$\Rightarrow \quad x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$$

29. Soln. We have,

$$x\frac{dy}{dx} + y - x + xy\cot x = 0, \ (x \neq 0)$$

$$\Rightarrow \quad x\frac{dy}{dx} + (1 + x\cot x).y = x$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{1 + x\cot x}{x}.y = 1 \qquad ....(i)$$

This is linear D.E. of the form $\frac{dy}{dx} + Py = Q$

Where $P = \frac{1 + x\cot x}{x} = \frac{1}{x} + \cot x$ and $Q = 1$

$\therefore$ Now $I.F. = e^{\int Pdx} = e^{\log(x\sin x)} = x\sin x$

$\therefore$ The solution of (i) is $y.x\sin x = \int 1.x\sin x\,dx + C$

$= x(-\cos x) + \int 1.\cos x dx + C.$

$\Rightarrow xy\sin x = -x\cos x + \sin x + C$

The required solution

$y.x\sin x = x(-\cos x) + \sin x + C$

..........(i)

Putting $x = \frac{\pi}{2}, y = 0$ in (ii), we get

$$\therefore \quad 0 = -\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + C \Rightarrow C = -1$$

30. Soln. We have,

$$\left(1 + y^2\right) + \left(x - e^{\tan^{-1} y}\right)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \left(x - e^{\tan^{-1} y}\right)\frac{dy}{dx} = -\left(1 + y^2\right)$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{x - e^{\tan^{-1} y}}{-(1 + y^2)} \Rightarrow \frac{dx}{dy} + \frac{1}{1 + y^2}.x = \frac{e^{\tan^{-1} y}}{1 + y^2}$$

This is a linear differential equation of the form

$\frac{dx}{dy} + Px = Q$, where

$$P = \frac{1}{1 + y^2} \text{ and } Q = \frac{e^{\tan^{-1} y}}{1 + y^2}$$

$$\therefore \quad I.F. = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1} y}$$

$\therefore$ Solution is

$$x.e^{\tan^{-1} y} = \int \frac{\left(e^{\tan^{-1} y}\right)^2}{1 + y^2} dy + C$$

$$= \quad \int \frac{e^{2\tan^{-1} y}}{1 + y^2} dy + C$$

$$\Rightarrow \quad x.e^{\tan^{-1} y} = \frac{e^{2\tan^{-1} y}}{2} + C_1$$

$$\Rightarrow \quad x = \frac{e^{\tan^{-1} y}}{2} + C_1 e^{-\tan^{-1} y}$$

SURE SHOT QUESTIONS

Chapter – 10 (Solution)

Vector Algebra

➢ MCQ (1 mark)

1. Soln. (c): Let $\vec{a}=\hat{i}-2\hat{j}+2\hat{k}$

$|\vec{a}|=\sqrt{1+4+4}=\sqrt{9}=3$

$\therefore$ Required vector = $\dfrac{9(\hat{i}-2\hat{j}+2\hat{k})}{3}=3(\hat{i}-2\hat{j}+2\hat{k})$

2. Soln. (d): Given points are $2\vec{a}-3\vec{b}$ and $\vec{a}+\vec{b}$

Ratio = 3 : 1

$\therefore$ Required vector = $\dfrac{(2\vec{a}-3\vec{b})\times1+(\vec{a}+\vec{b})\times3}{3+1}$

$=\dfrac{2\vec{a}-3\vec{b}+3\vec{a}+3\vec{b}}{4}=\dfrac{5}{4}\vec{a}$

3. Soln. (c): Let A(2, 5, 0) and B(-3, 7, 4)

$\therefore$ Required vector = $(-3-2)\hat{i}+(7-5)\hat{j}+(4-0)\hat{k}$

$=-5\hat{i}+2\hat{j}+4\hat{k}$

4. Soln. (b): $\vec{a}.\vec{b}=2\sqrt{3},\ |\vec{a}|=\sqrt{3},\ |\vec{b}|=4$

Let θ be the angle between $\vec{a}$ and $\vec{b}$

$\therefore\quad \vec{a}.\vec{b}=|\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow\quad 2\sqrt{3}=\sqrt{3}.4.\cos\theta \Rightarrow \cos\theta=\dfrac{1}{2}\Rightarrow\theta=\dfrac{\pi}{3}$

5. Soln. (d): We have, $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$,

$\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$ are orthogonal

$\therefore\quad \vec{a}.\vec{b}=0\Rightarrow(2\hat{i}+\lambda\hat{j}+i=\hat{k}).(\hat{i}+2\hat{j}+3\hat{k})=0$

$\Rightarrow 2.1+\lambda.2+1.3=0\Rightarrow\lambda=-\dfrac{5}{2}$

6. Soln. (a): $\vec{a}=3\hat{i}-6\hat{j}+\hat{k}$ and $\vec{b}=2\hat{i}-4\hat{j}+\lambda\hat{k}$

Since, $\vec{a}$ and $\vec{b}$ are parallel. $\quad\therefore\ \vec{a}\times\vec{b}=\vec{0}$

$$\Rightarrow\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -6 & 1\\ 2 & -4 & \lambda\end{vmatrix}=\vec{0}$$

$\Rightarrow(-6\lambda+4)\hat{i}-(3\lambda-2)\hat{j}+(-12+12)\hat{k}=\vec{0}$

$\Rightarrow(-6\lambda+4)\hat{i}+(2-3\lambda)\hat{j}=0\hat{i}+0\hat{j}$

Comparing coefficients of $\hat{i}$ and $\hat{j}$, we get

$$-6\lambda+4=0\ and\ 2-3\lambda=0\Rightarrow\lambda=\frac{2}{3}$$

7. Soln. (d): $\vec{a}=2\hat{i}-3\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}+3\hat{j}+\hat{k}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -3 & 2\\ 2 & 3 & 1\end{vmatrix}=-9\hat{i}+2\hat{j}+12\hat{k}$$

Area of

$\Delta OAB=\dfrac{1}{2}|\vec{a}\times\vec{b}|=\dfrac{1}{2}\sqrt{81+4+144}=\dfrac{1}{2}\sqrt{299}$

8. Soln. (d): Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

Then, $\quad|\vec{a}|=\sqrt{a_1^2+a_2^2+a_3^2}$

Since, $(\vec{a}\times\hat{i}).(\vec{a}\times\hat{i}) = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\hat{i} \\ \hat{i}.\vec{a} & 1 \end{vmatrix} = |\vec{a}|^2 - a_1^2$

Similarly, $(\vec{a}\times\hat{j})^2 = |\vec{a}|^2 - a_2^2$ and $(\vec{a}\times\hat{k})^2 = |\vec{a}|^2 - a_3^2$

$\therefore\ (\vec{a}\times\hat{i})^2 + (\vec{a}\times\hat{j})^2 + (\vec{a}\times\hat{k})^2$

$= 3|\vec{a}|^2 - (a_1^2 + a_2^2 + a_3^2) = 3|\vec{a}|^2 - |\vec{a}|^2 = 2\vec{a}^2$

9. Soln. (d): $|\vec{a}| = 10,\ |\vec{b}| = 2,\ \vec{a}.\vec{b} = 12$

We know, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow 12 = 10\times 2\cos\theta \Rightarrow \cos\theta = \frac{3}{5}$

$\therefore \quad \sin\theta = \frac{4}{5}$

Now, $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta = 10\times 2\times\frac{4}{5} = 16$

10. Soln. (a): Since, given vectors are coplanar

$$\begin{vmatrix} \lambda & 1 & 2 \\ 1 & \lambda & -1 \\ 2 & -1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \quad \lambda(\lambda^2 - 1) - 1(\lambda + 2) + 2(-1 - 2\lambda) = 0$

$\Rightarrow \quad \lambda^3 - \lambda - \lambda - 2 - 2 - 4\lambda = 0 \Rightarrow \lambda^3 - 6\lambda - 4 = 0$

$\Rightarrow \quad (\lambda + 2)(\lambda^2 - 2\lambda - 2) = 0$

$\Rightarrow \quad \lambda + 2 = 0 \ or\ \lambda^2 - 2\lambda - 2 = 0$

$\Rightarrow \quad \lambda = -2 \ or\ \lambda = \frac{2\pm\sqrt{12}}{2} \Rightarrow \lambda = -2,\ 1\pm\sqrt{3}$

11. Soln. (c): $\vec{a} + \vec{b} + \vec{c} = \vec{0} \Rightarrow (\vec{a} + \vec{b} + \vec{c})^2 = 0$

$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = 0$

$\Rightarrow \quad (1+1+1) + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = 0$

$\Rightarrow \quad \vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} = -\frac{3}{2}$

12. Soln. (b): Projection of $\vec{a}$ and $\vec{b}$ is $\frac{\vec{a}.\vec{b}}{|\vec{b}|}$

13. Soln. (c): $\vec{a}.\vec{b} + \vec{c} = \vec{0} \Rightarrow (\vec{a} + \vec{b} + \vec{c})^2 = 0$

$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = 0$

$\Rightarrow \quad (2)^2 + (3)^2 + (5)^2 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = 0$

$\Rightarrow \quad \vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} = \frac{-1}{2}[4 + 9 + 25] = \frac{-1}{2}\times 38 = -19$

14. Soln. (c): $-3 \le \lambda \le 2 \Rightarrow |\lambda| \le 3$

Now, $\quad |\lambda||\vec{a}| \le 3|\vec{a}| \Rightarrow |\lambda\vec{a}| \le 12$

$\therefore \quad$ Range of $|\lambda\vec{a}|$ is [0, 12]

15. Soln. (b): Let $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ be the unit vector which is perpendicular to both $\vec{a}$ and $\vec{b}$, then

$\vec{a}.\vec{c} = 0 \Rightarrow 2c_1 + c_2 + 2c_3 = 0$ ....(i)

And $\vec{b}.\vec{c} = 0 \Rightarrow c_2 + c_3 = 0$ ....(ii)

$\because \vec{c}$ is a unit vector

$\Rightarrow \quad c_1^2 + c_2^2 + c_3^2 = 1$ ....(iii)

From (i) and (ii), we get

$2c_1 + c_3 = 0$ ....(iv)

From (ii) and (iii), we get

$c_1^2 + 2c_3^2 = 1$ ....(v)

Now, from (iv) and (v)

$c_1^2 + 2(-2c_1)^2 = 1 \Rightarrow c_1^2 + 8c_1^2 = 1$

$\Rightarrow \quad c_1^2 = \frac{1}{9} \Rightarrow c_1 = \pm\frac{1}{3}$

Putting the value of c_1 in (iv), we get

$c_3 = \mp\frac{2}{3}$

Now, from (i)

$c_2 = -2(c_1 + c_3)$

If $c_1 = \frac{1}{3}$, then $c_3 = \frac{-2}{3}$

$$\therefore \quad c_2 = -2\left[\frac{1}{3}+\left(\frac{-2}{3}\right)\right] = -2\left[\frac{-1}{3}\right] = \frac{2}{3}$$

If $c_1 = -\frac{1}{3}$, *then* $c_3 = \frac{2}{3}$

$$\therefore \quad c_2 = -2\left[\frac{-1}{3}+\frac{2}{3}\right] = -2\left[\frac{1}{3}\right] = \frac{-2}{3}$$

Hence, vector $\vec{c} = \frac{1}{3}\hat{i}+\frac{2}{3}\hat{j}-\frac{2}{3}\hat{k}$ and

$\vec{c} = -\frac{1}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}$

Hence, there are two unit vectors perpendicular to the given vectors $\vec{a}$ and $\vec{b}$.

16. Ans. (b)

17. Soln.

(b): Let $\vec{a} = (\hat{i}+\hat{j}+\hat{k})$

So, unit vector of $\vec{a} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$

$\therefore$ The value of p is $\frac{1}{\sqrt{3}}$.

18. Soln. (a) :

$$\overrightarrow{EA}+\overrightarrow{EB}+\overrightarrow{EC}+\overrightarrow{ED} = \overrightarrow{EA}+\overrightarrow{EB}-\overrightarrow{EA}-\overrightarrow{EB}$$

[As diagonals of a rhombus bisect each other]

$= \vec{0}$

19. Soln.

(b): Let $\vec{v} = 4\hat{i}-3\hat{k}$

$$\therefore \ |\vec{v}| = \sqrt{4^2+(3)^2} = \sqrt{16+9} = \sqrt{25} = 5$$

Now, $\hat{v}$ = unit vector along $\vec{v}$

$$= \frac{\vec{v}}{|\vec{v}|} = \frac{1}{5}(4\hat{i}-3\hat{k})$$

20. Soln.

(b): Given, $\vec{a}.\vec{b} \geq 0$

$\Rightarrow |\vec{a}||\vec{b}|\cos\theta \geq 0$

Assuming $|\vec{a}| \neq 0$ and $|\vec{b}| \neq 0$

$\Rightarrow \cos\theta \geq 0 \ [\because |\vec{a}| \geq 0 \, |\vec{b}| \geq 0] \Rightarrow \theta \in \left[0, \frac{\pi}{2}\right]$

21. Soln. (c): Given vector is $6\hat{i}-2\hat{j}+3\hat{k}$

$\therefore$ Its magnitude = $\sqrt{6^2+(-2)^2+3^2}$

$= \sqrt{36+4+9} = \sqrt{49} = 7$ units

22. Soln. (c): Here, $\vec{a} = \hat{i}-2\hat{j}+3\hat{k}$, $\vec{b} = 2\hat{i}+\lambda\hat{k}$

Since, projection of $\vec{a}$ on $\vec{b}$ = 0

$$\Rightarrow \frac{\vec{a}.\vec{b}}{|\vec{b}|} = 0 \Rightarrow \frac{(\hat{i}-2\hat{j}+3\hat{k}).(2\hat{i}+\lambda\hat{k})}{\sqrt{2^2+\lambda^2}} = 0$$

$$\Rightarrow \frac{2+3\lambda}{\sqrt{4+\lambda^2}} = 0 \Rightarrow 2+3\lambda = 0 \Rightarrow \lambda = -\frac{2}{3}$$

23. Soln. (c): Since, $\hat{i}, \hat{j}, \hat{k}$ are mutually perpendicular to each other.

$\therefore \hat{i}.\hat{k} = 0$

➤ Assertion-Reasoning (1 mark)

24. **Sol. (d)** A is false but R is true.

Explanation: Assertion (A) is wrong. The position of a particle in a rectangular coordinate system is (3, 2, 5). Then its position vector be

$3\hat{i} + 2\hat{j} + 5\hat{k}$

Reason (R) is correct. The displacement vector of the particle that moves from point P(2, 3, 5) to point Q(3, 4, 5)

$= (3 - 2)\hat{i} + (4 - 3)\hat{j} + (5 - 5)\hat{k}$

$= \hat{i} + \hat{j}$

25. Sol. **(b)** Both A and R are true but R is not the correct explanation of A.

Explanation: We can multiply any vector by any scalar. For example in equation $\vec{F} = m\vec{a}$

mass is a scalar quantity, but acceleration is a vector quantity.

26. Sol. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation: Cross product of two vectors is perpendicular to the plane containing both the vectors.

27. Sol. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation: Both A and R are true and R is the correct explanation of A.

28. Sol. **(b)** Both A and R are true but R is not the correct explanation of A.

Explanation: $|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$

gives = $\theta = 90^{\circ}$|

Also vector addition is commutative.

➢ Case Study Questions

29. **Sol. (d)** (3, 2, 3)

Explanation: Clearly, G be the centroid of BCD, therefore coordinates of G are

$$\left(\frac{3+4+2}{3}, \frac{0+3+3}{3}, \frac{1+6+2}{3}\right) = (3, 2, 3)$$

(ii) **(a)** $\sqrt{11}$ units

Explanation: Since, A ≡ (0, 1, 2) and G = (3, 2, 3)

$\therefore \vec{AG} = (3-0)\hat{i} + (2-1)\hat{j} + (3-2)\hat{k} = 3\hat{i} + \hat{j} + \hat{k}$

$\Rightarrow |\vec{AG}|^2 = 3^2 + 1^2 + 1^2 = 9 + 1 + 1 = 11$

$\Rightarrow |\vec{AG}| = \sqrt{11}$

(iii) **(a)** $3\sqrt{10}$

Explanation: Clearly, area of $\triangle ABC = \frac{1}{2}|\vec{AB} \times \vec{AC}|$

Here, $\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3-0 & 0-1 & 1-2 \\ 4-0 & 3-1 & 6-2 \end{vmatrix}$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & -1 \\ 4 & 2 & 4 \end{vmatrix}$

$= \hat{i}(-4+2) - \hat{j}(12+4) + \hat{k}(6+4) = -2\hat{i} - 16\hat{j} + 10\hat{k}$

$\therefore |\vec{AB} \times \vec{AC}| = \sqrt{(-2)^2 + (-16)^2 + 10^2}$

$= \sqrt{4+256+100} = \sqrt{360} = 6\sqrt{10}$

Hence, area of $\triangle ABC = \frac{1}{2} \times 6\sqrt{10} = 3\sqrt{10}$ sq. units

(iv) **(b)** 9.32 units

Explanation: Here, $\vec{AB} = 3\hat{i} - \hat{j} - \hat{k}$

$\Rightarrow |\vec{AB}| = \sqrt{9+1+1} = \sqrt{11}$

Also, $\vec{AC} = 4\hat{i} + 2\hat{j} + 4\hat{k}$

$\Rightarrow |\vec{AC}| = \sqrt{16+4+16} = \sqrt{36} = 6$

Now, $|\vec{AB}| + |\vec{AC}| = \sqrt{11} + 6 = 3.32 + 6 = 9.32$ units

(v)

(c) $\frac{6}{\sqrt{10}}$ units

Explanation: The length of the perpendicular from the vertex D on the opposite face

$= |\text{Projection of } \vec{AD} \text{ on } \vec{AB} \times \vec{AC}|$

$= \left|\frac{(2\hat{i}+2\hat{j})\cdot(-2\hat{i}-16\hat{j}+10\hat{k})}{\sqrt{(-2)^2+(-16)^2+10^2}}\right|$

$= \left|\frac{-4-32}{\sqrt{360}}\right| = \frac{36}{6\sqrt{10}} = \frac{6}{\sqrt{10}}$ units

30. Sol.

(i) **(a)** 130 m/s

Explanation: 130 m/s

(ii) **(c)** $\tan^{-1}\left(\frac{5}{12}\right)$

Explanation: $\tan^{-1}\left(\frac{5}{12}\right)$

(iii) **(c)** 468 km
Explanation: 468 km

(iv) **(c)** 170 m/s
Explanation: 170 m/s

(v) **(b)** 612 km
Explanation: 612 km

➤ Questions

31. Soln.

We have, $\vec{a} = 4\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 1 \\ 2 & -2 & 1 \end{vmatrix}$$

$$= \hat{i}(-1+2) - \hat{j}(4-2) + \hat{k}(-8+2) = \hat{i} - 2\hat{j} - 6\hat{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{1^2 + (-2)^2 + (-6)^2} = \sqrt{41}$$

Unit vector along $\vec{a}\times\vec{b} = \dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

$$= \frac{\hat{i} - 2\hat{j} - 6\hat{k}}{\sqrt{41}} = \frac{1}{\sqrt{41}}\hat{i} - \frac{2}{\sqrt{41}}\hat{j} - \frac{6}{\sqrt{41}}\hat{k}$$

32. Soln.

Area of parallelogram = $\vec{a}\times\vec{b} = \left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{matrix}\right\|$

$$= [(-1+21)\hat{i} - (1-6)\hat{j} + (-7+2)\hat{k}]$$

$$= |20\hat{i} + 5\hat{j} - 5\hat{k}|$$

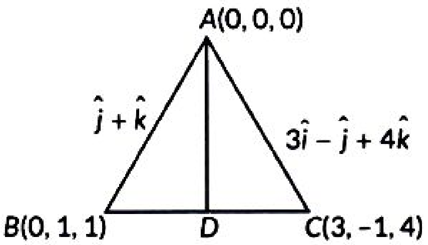
$$= \sqrt{(20)^2 + (5)^2 + (-5)^2}$$

$$= \sqrt{400+25+25} = \sqrt{450} = 15\sqrt{2}\ sq.units$$

33. Soln.

Take A to be as origin (0, 0, 0).

∴ Coordinates of B are (0, 1, 1) and coordinates of C are (3, -1, 4).



Let D be the mid point of BC and AD is a median of ΔABC.

∴ Coordinates of D are $\left(\frac{3}{2}, 0, \frac{5}{2}\right)$

So, length of AD = $\sqrt{\left(\frac{3}{2}-0\right)^2 + (0)^2 + \left(\frac{5}{2}-0\right)^2}$

$$= \sqrt{\frac{9}{4} + \frac{25}{4}} = \frac{\sqrt{34}}{2}\ \text{units}$$

34. Soln. We have,

$$\vec{a}+\vec{b} = (2\hat{i}+4\hat{j}-5\hat{k}) + (\hat{i}+2\hat{j}+3\hat{k})$$

$$= 3\hat{i} + 6\hat{j} - 2\hat{k}$$

$$\Rightarrow |\vec{a}+\vec{b}| = \sqrt{(3)^2 + (6)^2 + (-2)^2}$$

$$= \sqrt{(9+36+4)}$$

$$= \sqrt{49} = 7$$

Hence, the unit vector in the direction of $\vec{a}+\vec{b}$

$$= \frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}$$

$$= \frac{1}{7}(3\hat{i} + 6\hat{j} - 2\hat{k})$$

$$= \frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}$$

35. Soln. Let $\dot{a}$, $\dot{b}$, $\dot{c}$ and $\dot{d}$ be the position vectors of the vertices of the quadrilateral ABCD relative to some chosen point O as the origin. Since, E and F are the mid – points of the sides AC and BD respectively, so the position vectors of E and F are

$$\frac{\vec{a}+\vec{c}}{2}\ and\ \frac{\vec{b}+\vec{d}}{2}$$

We have, $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD}$

$= (\vec{b}-\vec{a})+(\vec{d}-\vec{a})+(\vec{b}-\vec{c})+(\vec{d}-\vec{c})$

$= 2(\vec{b}+\vec{d}-\vec{c}-\vec{a})$

$= 4\left(\dfrac{\vec{b}+\vec{d}}{2}\right)-4\left(\dfrac{\vec{c}+\vec{a}}{2}\right)$

= 4 [Position vector of F – Position vector of E]

$= 4\overrightarrow{EF}$

36. Soln. Vertices of the given ΔABC are A(1,2,3)
B(2, -1, 4) and C(4, 5, -1)

$\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = (2\hat{i}-\hat{j}+4\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})$

$= \hat{i}-3\hat{j}+\hat{k}$

$\overrightarrow{AC} = \overrightarrow{OC}-\overrightarrow{OA} = (4\hat{i}+5\hat{j}-\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})$

$= 3\hat{i}+3\hat{j}-4\hat{k}$

$Required\ area = \dfrac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix} = 9\hat{i}+7\hat{j}+12\hat{k}$$

$|\overrightarrow{AB}\times\overrightarrow{AC}| = \sqrt{9^2+7^2+12^2}$

$= \sqrt{81+49+144} = \sqrt{274}$

$\therefore$ Area of $\Delta ABC = \dfrac{1}{2}\sqrt{274}$ sq. units

Ans.

37. Soln. Let A and B be the given points with position vectors $3\vec{a}-2\vec{b}$ and $2\vec{a}+3\vec{b}$ respectively.
Let P and Q be the points dividing AB in the ratio 2 : 1 internally and externally respectively. Then,

Position vector of P = $\dfrac{1(3\vec{a}-2\vec{b})+2(2\vec{a}+3\vec{b})}{1+2}$

$= \dfrac{7\vec{a}}{3}+\dfrac{4\vec{b}}{3}$

Position vector of $Q = \dfrac{1(3\vec{a}-2\vec{b})-2\left(2\vec{a}+3\vec{b}\right)}{1-2}$

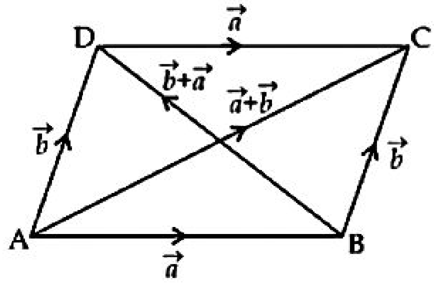
$= \vec{a}+8\vec{b}$

Ans.

38. Soln. Let ABCD be a parallelogram such that

$\overrightarrow{AB} = \vec{a} = 2\hat{i}-4\hat{j}-5\hat{k}$

And $\overrightarrow{BC} = \vec{b} = 2\hat{i}+2\hat{j}+3\hat{k}$



Then,

$\overrightarrow{AB}+\overrightarrow{BC} = \overrightarrow{AC}$

$\Rightarrow \quad \overrightarrow{AC} = \vec{a}+\vec{b} = 4\hat{i}-2\hat{j}-2\hat{k}$

$and \quad \overrightarrow{AB}+\overrightarrow{BD} = \overrightarrow{AD}$

$\Rightarrow \quad \overrightarrow{BD} = \overrightarrow{AD}-\overrightarrow{AB}$

$\Rightarrow \quad \overrightarrow{BD} = \vec{b}-\vec{a} = 0\hat{i}+6\hat{j}+8\hat{k}$

$Now, \quad \overrightarrow{AC} = 4\hat{i}-2\hat{j}-2\hat{k}$

$|\overrightarrow{AC}| = \sqrt{(4)^2+(-2)^2+(-2)^2} = \sqrt{24} = 2\sqrt{6}$

$and \quad |\overrightarrow{BD}| = 0\hat{i}+6\hat{j}+8\hat{k}$

$|\overrightarrow{BD}| = \sqrt{(0)^2+(6)^2+(8)^2} = \sqrt{100} = 10$

Unit vector along $\overrightarrow{AC}$

$= \dfrac{\overrightarrow{AC}}{|\overrightarrow{AC}|} = \dfrac{1}{2\sqrt{6}}(4\hat{i}-2\hat{j}-2\hat{k})$

$= \dfrac{1}{\sqrt{6}}(2\hat{i}-\hat{j}-\hat{k})$

Unit vector along $\overrightarrow{BD}$

$= \dfrac{\overrightarrow{BD}}{|\overrightarrow{BD}|} = \dfrac{1}{10}(6\hat{j}+8\hat{k}) = \dfrac{1}{5}(3\hat{j}+4\hat{k})$

Now, area of parallelogram

$= \dfrac{1}{2}|\overrightarrow{AC}\times\overrightarrow{BD}|$

$$\Rightarrow \quad \overrightarrow{AC}\times\overrightarrow{BD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & -2 \\ 0 & 6 & 8 \end{vmatrix}$$

$$= 4\hat{i} + 32\hat{j} + 24\hat{k}$$

Area of parallelogram

$$= \frac{1}{2}\left|\overrightarrow{AC}\times\overrightarrow{BD}\right|$$

$$= 2\sqrt{101} \ \ sq.\ units \qquad Ans.$$

39. Soln. $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

Unit vector in the direction of

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{3\hat{i}-2\hat{j}+6\hat{k}}{\sqrt{3^2+(-2)^2+6^2}} = \frac{1}{7}(3\hat{i}-2\hat{j}+6\hat{k})$$

40. Soln. Let $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + \hat{j}$

Projection of

$$\vec{a} \ on \ \vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{(\hat{i}-\hat{j}).(\hat{i}+\hat{j})}{\sqrt{1+1}} = \frac{1-1}{\sqrt{2}} = 0$$

41. Soln. $\vec{a}.\vec{b} = 8$

$\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$

We know projection of $\vec{a}$ on $\vec{b}$ =

$$\frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{8}{\sqrt{4+36+9}} = \frac{8}{7}$$

42. Soln. Projection of $\vec{a}$ on $\vec{b}$ = 4, (given)

$$\frac{\vec{a}.\vec{b}}{|\vec{b}|} = 4$$

$\therefore$

$$or \quad \frac{(\lambda\hat{i}+\hat{j}+4\hat{k}).(2\hat{i}+6\hat{j}+3\hat{k})}{\sqrt{4+36+9}} = 4$$

$$or \quad 2\lambda + 6 + 12 = 7\times 4$$

$$or \quad 2\lambda = 28 - 18 = 10$$

$$or \quad \lambda = \frac{10}{2} = 5.$$

43. Soln. Given, $|\vec{a}+\vec{b}| = |\vec{a}|$

$$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2$$

Or $|\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = |\vec{a}|^2$

$$|\vec{b}|^2 + 2\vec{a}.\vec{b} = 0 \qquad ....(i)$$

Now, $(2\vec{a}+\vec{b}).\vec{b} = 2\vec{a}.\vec{b} + \vec{b}.\vec{b}$

$$\vec{b}.(2\vec{a}.\vec{b}+\vec{b}) = 0$$

From eqn. (i), $2\vec{a}+\vec{b}\ is \perp \ to \ \vec{b}$.

44. Soln. We know that, the direction ratio's a, b, c of a vector $\vec{r} = x\hat{i} + y\hat{i} + z\hat{k}$ are just the respective components x, y and z of the vector. So, for the given vector, we have a = 1, b = 1 and c = -2 Further, if l, m and n are the direction cosines of the given vector, then

$$l = \frac{a}{|\vec{r}|} = \frac{1}{\sqrt{6}},\ m = \frac{b}{|\vec{r}|} = \frac{1}{\sqrt{6}},\ n = \frac{c}{|\vec{r}|} = \frac{-2}{\sqrt{6}} \ as\ |\vec{r}| = \sqrt{6}$$

Thus the direction cosines are $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$.

45. Soln. Any vector perpendicular to both $\vec{\alpha}$ and $\vec{\beta}$ = Parallel to $(\vec{\alpha}\times\vec{\beta})$

$\therefore \quad \vec{p} = \lambda(\vec{\alpha} \times \vec{\beta})$

$$= \lambda \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 5 & -1 \\ 1 & -4 & 5 \end{vmatrix}$$

$= \lambda[\hat{i}(25-4) - \hat{j}(20+1) + \hat{k}(-16-5)]$

$= \lambda[21\hat{i} - 21\hat{j} - 21\hat{k}]$

p.q = 21 (Given)

$\lambda(21\hat{i} - 21\hat{j} - 21\hat{k}).(3\hat{i} + \hat{j} - \hat{k}) = 21$

$\lambda(63 - 21 + 21) = 21$

$\lambda = \frac{1}{3}$

$\therefore \quad p = \lambda(21\hat{i} - 21\hat{j} - 21\hat{k})$

$p = \frac{1}{3}(21\hat{i} - 21\hat{j} - 21\hat{k})$

$p = 7\hat{i} - 7\hat{j} - 7\hat{k}$

46. Soln. Given,

$\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$

And $\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$

$\vec{a} + \vec{b} = (\hat{i} - \hat{j} + 7\hat{k}) + (5\hat{i} - \hat{j} + \lambda\hat{k})$

$= 6\hat{i} - 2\hat{j} + (7+\lambda)\hat{k}$

And $\vec{a} - \vec{b} = (\hat{i} - \hat{j} + 7\hat{k}) - (5\hat{i} - \hat{j} + \lambda\hat{k})$

$= -4\hat{i} + (7-\lambda)\hat{k}$

Since $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are perpendicular vectors,

$\therefore \quad (\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = 0$

$\{6\hat{i} - 2\hat{j} + (7+\lambda)\hat{k}\}.\{-4\hat{i} + (7-\lambda)\hat{k}\} = 0$

or $-24 + (7+\lambda)(7-\lambda) = 0$

or $49 - \lambda^2 - 24 = 0$

or $\lambda^2 = 49 - 24 = 25$

or $\lambda = \pm 5\, units$

47. Soln. Since $\vec{a} \perp (\vec{b} + \vec{c})$, therefore

$\vec{a}.(\vec{b} + \vec{c}) = 0$

Or $\vec{a}.\vec{b} + \vec{a}.\vec{c} = 0$

Or $\vec{a}.\vec{a} + (\vec{a}.\vec{b} + \vec{a}.\vec{c}) = \vec{a}.\vec{a} + 0$

Or $\vec{a}.(\vec{a} + \vec{b} + \vec{c}) = |\vec{a}|^2$

Or $\vec{a}.(\vec{a} + \vec{b} + \vec{c}) = 3^2 = 9$ ....(i)

Similarly, $\vec{b}.(\vec{a} + \vec{b} + \vec{c}) = |\vec{b}|^2 = 16$ ....(ii)

And $\vec{c}.(\vec{a} + \vec{b} + \vec{c}) = |\vec{c}|^2 = 25$ ....(iii)

Adding eqn. (i), (ii) and (iii), we get

$\vec{a}.(\vec{a} + \vec{b} + \vec{c}) + \vec{b}.(\vec{a} + \vec{b} + \vec{c}) + \vec{c}.(\vec{a} + \vec{b} + \vec{c}) = 50$

or $(\vec{a} + \vec{b} + \vec{c}).(\vec{a} + \vec{b} + \vec{c}) = 50$

or $|\vec{a} + \vec{b} + \vec{c}|^2 = 50$

or $|\vec{a} + \vec{b} + \vec{c}| = \sqrt{50} = 5\sqrt{2}$

48. Soln. Given, $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$,

$\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$

Also, $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$.

$\therefore \quad (\vec{a} + \lambda\vec{b}).\vec{c} = 0$ ....(i)

$[\because$ when $\vec{a} \perp \vec{b}$, then $\vec{a}.\vec{b} = 0]$

Now,

$\vec{a} + \lambda\vec{b} = (2\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} + 2\hat{j} + \hat{k})$

Or

$\vec{a} + \lambda\vec{b} = \hat{i}(2-\lambda) + \hat{j}(2+2\lambda) + \hat{k}(3+\lambda)$

Then from Eq. (i), we get

$[\hat{i}(2-\lambda) + \hat{j}(2+2\lambda) + \hat{k}(3+\lambda)].[3\hat{i} + \hat{j}] = 0$

or $3(2-\lambda) + 1(2+2\lambda) = 0$

or $8 - \lambda = 0$

$\therefore \quad \lambda = 8$

49. Soln. $\overrightarrow{BA} = 2\hat{i} + 4\hat{j} - 4\hat{k}$

Or d-ratios of $\overrightarrow{BA}$ are 2, 4, -4

$\therefore$ Direction cosines are: $\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}$

50. Soln. $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 2 & -1 & -1 \end{vmatrix}$

$$= i(1+2) - j(-1-4) + k(-1+2)$$
$$= 3i + 5j + k$$

Area of $|\vec{a}\times\vec{b}| = \sqrt{9+25+1}$

$= \sqrt{35}$ sq. units

51. Soln. Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$

Given $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

$\vec{b} = \hat{j} - \hat{k}$

According to the question,

$$\vec{a}.\vec{c} = 3$$

Or $(\hat{i} + \hat{j} + \hat{k}).(x\hat{i} + y\hat{j} + z\hat{k}) = 3$

Or $x + y + z = 3$

..........(i)

And $\vec{a}\times\vec{c} = \vec{b}$

Or $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \vec{b} = \hat{j} - \hat{k}$

Or $(z-y)\hat{i} + (x-z)\hat{j} + (y-x)\hat{k} = \hat{j} - \hat{k}$

On equating the coefficients of like terms, we get

$z - y = 0, \; or \; y = z$

............(ii)

$x - z = 1$

..........(iii)

And $y - x = -1$ ....(iv)

Solving eqns. (i), (ii), (iii) and (iv), we get

$x = 5/3$

And $y = 2/3 = z$

Hence, $\vec{c} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

52. Soln. Given $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

$\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$

$3\vec{b} = 3\hat{i} + 3\hat{j} + 3\hat{k}$

And $2\vec{b} = 2\hat{i} + 4\hat{j} + 6\hat{k}$

Let $\vec{c}$ be the vector perpendicular to both $(3\vec{a} + 2\vec{b})$ and $(3\vec{a} - 2\vec{b})$.

$\therefore$ $3\vec{a} + 2\vec{b} = 5\hat{i} + 7\hat{j} + 9\hat{k}$ and

$3\vec{a} - 2\vec{b} = \hat{i} - \hat{j} - 3\hat{k}$

$\therefore$ $\vec{c} = (3\vec{a} + 2\vec{b})\times(3\vec{a} - 2\vec{b})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 7 & 9 \\ 1 & -1 & -3 \end{vmatrix}$$

$$= \hat{i}(-21+9) - \hat{j}(-15-9) + \hat{i}(-5-7)$$

$$= -12\hat{i} + 24\hat{j} - 12\hat{k}$$

53. Soln. Let $x\hat{i} + y\hat{j} + z\hat{k}$ be the unit vector along $\vec{c}$. Since $-\hat{i} + \hat{j} - \hat{k}$ bisects the angle between $\vec{c}$ and $3\hat{i} + 4\hat{j}$.

Therefore,

$$\lambda(-\hat{i} + \hat{j} - \hat{k}) = (x\hat{i} + y\hat{j} + z\hat{k}) + \frac{3\hat{i} + 4\hat{j}}{5}$$

$$x + \frac{3}{5} = -\lambda$$

$$y + \frac{4}{5} = -\lambda$$

And $z = -\lambda$

Now, $x^2 + y^2 + z^2 = 1$

$$\left[\because x\hat{i} + y\hat{j} + z\hat{k} \; is\, a\, unit\, vector\right]$$

Or $\left(-\lambda - \frac{3}{5}\right)^2 + \left(\lambda - \frac{4}{5}\right)^2 + \lambda^2 = 1$

Or $3\lambda^2 - \frac{2}{5}\lambda = 0$

Or $\lambda = 0 \; or \; \lambda = \frac{2}{15}$

But $\lambda \neq 0$, because $\lambda = 0$ implied that the given vectors are parallel.

Or $\lambda = -\frac{2}{15}$,

$y = \frac{-10}{15}$

And $z = \frac{-2}{15}$

Hence,

$$x\hat{i}+y\hat{j}+z\hat{k}=-\frac{1}{15}(11\hat{i}+10\hat{j}+2\hat{k})$$

54. Soln. Let angle between $\vec{a}$ and $\vec{b}$ be θ.

Given, $(\vec{a}+3\vec{b})\perp(7\vec{a}-5\vec{b})$

Or $(\vec{a}+3\vec{b}).(7\vec{a}-5\vec{b})=0$

Or $7|\vec{a}|^2+16(\vec{a}.\vec{b})-15|\vec{b}|^2=0$

Or $7+16\cos\theta-15=0$

$$\left[\because |\vec{a}|^2=|\vec{b}|^2=1\right]$$

Or $\cos\theta=\frac{8}{16}=\frac{1}{2}$

Or $\theta=\frac{\pi}{3}$

Also, given that $(\vec{a}-4\vec{b})\perp(7\vec{a}-2\vec{b})$

Or $(\vec{a}-4\vec{b}).(7\vec{a}-2\vec{b})=0$

Or $7|\vec{a}|^2+8|\vec{b}|^2-30(\vec{a}.\vec{b})=0$

Or $15-30\cos\theta=0$

Or $\cos\theta=\frac{1}{2}$

Or $\theta=\frac{\pi}{3}$

55. Soln. Given, ΔABC with vertices

$A(1,2,3)=\hat{i}+2\hat{j}+3\hat{k},\ B(2,-1,4)=2\hat{i}-\hat{j}+4\hat{k},$

$C(4,5,-1)=4\hat{i}+5\hat{j}-\hat{k}$

Now

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(2\hat{i}-\hat{j}+4\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})$$

$$=\hat{i}-3\hat{j}+\hat{k}.$$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(4\hat{i}+5\hat{j}-\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})$$

$$=3\hat{i}+3\hat{j}-4\hat{k}$$

$$\therefore\ (\overrightarrow{AB}\times\overrightarrow{AC})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 1\\ 3 & 3 & -4\end{vmatrix}=9\hat{i}+7\hat{j}+12$$

Hence, area of $\Delta ABC=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

$$=\frac{1}{2}|9\hat{i}+7\hat{j}+12\hat{k}|$$

$$=\frac{1}{2}\sqrt{9^2+7^2+12^2}=\frac{1}{2}\sqrt{81+49+144}$$

$$=\frac{1}{2}\sqrt{274}\ sq.\ units$$

56. Soln. Given, position vector of $A=\hat{i}+\hat{j}+\hat{k}$

Position vector of $B=2\hat{i}+5\hat{j}$

Position vector of $C=3\hat{i}+2\hat{j}-3\hat{k}$

Position vector of $D=\hat{i}-6\hat{j}-\hat{k}$

$$\therefore\quad \overrightarrow{AB}=(2\hat{i}+5\hat{j})-(\hat{i}+\hat{j}+\hat{k})=\hat{i}+4\hat{j}-\hat{k}$$

And

$$\overrightarrow{CD}=(\hat{i}-6\hat{j}-\hat{k})-(3\hat{i}+2\hat{j}-3\hat{k})=-2\hat{i}-8\hat{j}+2\hat{k}$$

Now, $|\overrightarrow{AB}|=\sqrt{(1)^2+(4)^2+(-1)^2}=\sqrt{18}$

$$|\overrightarrow{CD}|=\sqrt{(-2)^2+(-8)^2+(2)^2}=\sqrt{4+64+4}$$

$$=\sqrt{72}=2\sqrt{18}$$

Let θ be the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

$$\therefore\ \cos\theta=\frac{\overrightarrow{AB}.\overrightarrow{CD}}{|\overrightarrow{AB}||\overrightarrow{CD}|}=\frac{(\hat{i}+4\hat{j}-\hat{k}).(-2\hat{i}-8\hat{j}+2\hat{k})}{(\sqrt{18})(2\sqrt{18})}$$

$$=\frac{-2-32-2}{36}=\frac{-36}{36}=-1$$

$$\Rightarrow\quad \cos\theta=-1\ \Rightarrow\ \theta=\pi$$

Since, angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is 180^0.

$\therefore\ \overrightarrow{AB}$ *and* $\overrightarrow{CD}$ are collinear.

57. Soln. Let $\vec{d}=x\hat{i}+y\hat{j}+z\hat{k}$

Now, it is given that, $\vec{d}$ is perpendicular to

$\vec{b}=\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}-\hat{k}$

$$\therefore\quad \vec{d}.\vec{b}=0\ \ and\ \ \vec{d}.\vec{c}=0$$

$\Rightarrow \quad x-4y+5z=0$ ....(i)

And $3x+y-z=0$ ......(ii)

Also, $\vec{d}.\vec{a}=21$, where $\vec{a}=4\hat{i}+5\hat{j}-\hat{k}$

$\Rightarrow \quad 4x+5y-z=21$ ....(iii)

Eliminating z from (i) and (ii), we get

$16x+y=0$ ....(iv)

Eliminating z from (ii) and (iii), we get

$x+4y=21$ ....(v)

Solving (iv) and (v), we get

$$x=\frac{-1}{3}, y=\frac{16}{3}$$

Putting the values of x and y in (i), we get $z=\frac{13}{3}$

$\therefore \vec{d}=\frac{-1}{3}\hat{i}+\frac{16}{3}\hat{j}+\frac{13}{3}\hat{k}$ is the required vector.

58. Soln. Given $\vec{a}+\vec{b}+\vec{c}=0$ and

$|\vec{a}|=3, |\vec{b}|=5, |\vec{c}|=7$

We have $\vec{a}+\vec{b}+\vec{c}=0$

$\Rightarrow \quad \vec{a}+\vec{b}=-\vec{c} \Rightarrow |\vec{a}+\vec{b}|^2=|-\vec{c}|^2$

$\Rightarrow \quad |\vec{a}|^2+|\vec{b}|^2+2(\vec{a}.\vec{b})=|\vec{c}|^2$

$\Rightarrow \quad 9+25+2|\vec{a}||\vec{b}|\cos\theta=49$

$\Rightarrow \quad 2\times3\times5\times\cos\theta=49-34=15$

$\Rightarrow \quad \cos\theta=\frac{15}{30}=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}=60^0$

SURE SHOT QUESTIONS

ARVIND ACADEMY

Chapter – 11 (Solution)

Three-Dimensional Geometry

➢ MCQ (1 mark)

1. Soln. (d): Foot of perpendicular from (α,β,γ) on the y-axis is $(0,\beta,0)$.

$\therefore$ Distance of (α,β,γ) from y-axis = distance of (α,β,γ) from $(0,\beta,0)$

$$=\sqrt{(0-\alpha)^2+(\beta-\beta)^2+(0-\gamma)^2}=\sqrt{\alpha^2+\gamma^2}$$

2. Soln. (d): If (k, k, k) are direction cosines of a line, then

$$k^2+k^2+k^2=1 \Rightarrow 3k^2=1 \Rightarrow k^2=\frac{1}{3} \Rightarrow k=\pm\frac{1}{\sqrt{3}}$$

3. Soln. (a): $\vec{r}.\left(\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}-\frac{6}{7}\hat{k}\right)=1$

Here, $\vec{n}=\frac{2}{7}\hat{i}+\frac{3}{7}\hat{i}=j-\frac{6}{7}\hat{k}$

$$\therefore |\vec{n}|=\sqrt{\left(\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2+\left(\frac{-6}{7}\right)^2}=\sqrt{\frac{49}{49}}=1$$

So, $\vec{n}$ is a unit vector. Hence given equation of plane is in normal form.

$\therefore$ Distance from origin = 1

4. Soln. (d): DR's of the line is (3, 4, 5) and DR's of the normal to the plane is (2, -2, 1).

Let θ be the angle between line and plane, then $(90^0-\theta)$ be the angle between the line and normal of the plane. Then,

$$\cos(90^0-\theta)=\frac{3\times2+4\times(-2)+5\times1}{\sqrt{3^2+4^2+5^2}\sqrt{2^2+(-2)^2+1^2}}$$

$$\Rightarrow \sin\theta=\frac{6-8+5}{\sqrt{50}\sqrt{9}} \Rightarrow \sin\theta=\frac{3}{5\sqrt{2}\times3} \Rightarrow \sin\theta=\frac{\sqrt{2}}{10}$$

5. Soln. (d): Projection of $P(\alpha,\beta,\gamma)$ on xy-plane is $Q(\alpha,\beta,0)$.

If $P'(\alpha',\beta',\gamma')$ is the reflection of P in xy-plane, then Q is the mid – point of PP'

$$\Rightarrow (\alpha,\beta,0)=\left(\frac{\alpha+\alpha'}{2},\frac{\beta+\beta'}{2},\frac{\gamma+\gamma'}{2}\right)$$

$$\Rightarrow \frac{\alpha+\alpha'}{2}=\alpha,\ \frac{\beta+\beta'}{2}=\beta,\ \frac{\gamma+\gamma'}{2}=0$$

$$\Rightarrow \alpha'=\alpha,\ \beta'=\beta,\ \gamma'=-\gamma$$

$\therefore$ Required reflection is $(\alpha,\beta,-\gamma)$.

6. Soln. (a): Area of quadrilateral ABCD = Area of ΔABC + area of ΔACD

Now,

$$\overrightarrow{AB}=(2-0)\hat{i}+(3-4)\hat{j}+(-1-1)\hat{k}=2\hat{i}-\hat{j}-2\hat{k}$$

$$\overrightarrow{AC}=(4-0)\hat{i}+(5-4)\hat{j}+(0-1)\hat{k}=4\hat{i}+\hat{j}-\hat{k}$$

$$\therefore \overrightarrow{AB}\times\overrightarrow{AC}=\hat{i}(1+2)-\hat{j}(-2+8)+\hat{k}(2+4)$$

$$=3\hat{i}-6\hat{j}+6\hat{k}$$

$$\therefore Area\ of\ \Delta ABC=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$$

$$=\frac{1}{2}\sqrt{3^2+(-6)^2+6^2}=\frac{1}{2}\sqrt{9+36+36}=\frac{9}{2}$$

Similarly, as $\overrightarrow{AC}=4\hat{i}+\hat{j}-\hat{k}$ and $\overrightarrow{AD}=2\hat{i}+2\hat{j}+\hat{k}$

$$\therefore\ \overrightarrow{AC}\times\overrightarrow{AD}=3\hat{i}-6\hat{j}+6\hat{k}$$

$$\therefore\ Area\ of\ \Delta ACD=\frac{1}{2}|3\hat{i}-6\hat{i}+6\hat{k}|=\frac{1}{2}\sqrt{9+36+36}=\frac{9}{2}$$

$$\therefore\ Area\ of\ quadrilateral\ ABCD=\frac{9}{2}+\frac{9}{2}=9\ sq.units$$

7. Soln. (d): Given, xy + yz = 0

$$\Rightarrow\quad y(x+z)=0$$
$$\Rightarrow\quad y=0\ \ or\ \ x+z=0$$

$y=0$ is an equation of xz – plane

And x + z is also an equation of plane.

DR's of normal to the plane y = 0 are (0, 1, 0) and DR's of normal to the plane x + z = 0 are (1, 0, 1).

$$\because\ a_1a_2+b_1b_2+c_1c_2=0\times1+1\times0+0\times1=0$$

$\therefore$ Both planes are perpendicular.

8. Soln. (c): Direction ratios of x-axis are (1, 0, 0) and direction ratios of the normal to the plane $2x-3y+6z=11$ are (2, -3, 6).

$$\therefore\ \sin(\sin^{-1}\alpha)=\left|\frac{2+0+0}{\sqrt{0^2+0^2+1^2}\sqrt{2^2+(-3)^2+6^2}}\right|$$

$$\Rightarrow\quad \alpha=\left(\frac{2}{7}\right)$$

9. Soln.

(d): Given point is (p, q, r)

The foot of perpendicular drawn from point (p, q, r) on the y – axis is (0, q, 0).

Now, distance between these two points is

$$\sqrt{(p-0)^2+(q-p)^2+(r-0)^2}=\sqrt{p^2+r^2}$$

10. Soln. (c): Let P(4, -7, 3) be the given point and A be a point on y-axis s.t. $PA\perp$ to y-axis.

$$\therefore\quad A\equiv(0,-7,0)$$

Now, $PA=\sqrt{(4-0)^2+(-7-(-7))^2+(3-0)^2}$

$$=\sqrt{4^2+3^2}=\sqrt{16+9}=\sqrt{25}=5\text{ units}$$

11. Soln. (a): Vector equation of XY – plane is $\vec{r}.\hat{k}=0$.

12. Soln.

(d): Given that the direction cosines of a line are $\left(\frac{1}{u},\frac{1}{u},\frac{1}{u}\right)$.

We know that the sum of squares of the direction cosines is 1.

$$\Rightarrow\ \frac{1}{a^2}+\frac{1}{a^2}+\frac{1}{a^2}=1$$
$$\Rightarrow\ \frac{3}{a^2}=1\Rightarrow a^2=3$$
$$\Rightarrow\ a=\pm\sqrt{3}$$

13. Soln.

(a): Direction cosines are

$$\langle\cos 90^0,\cos 135^0,\cos 45^0\rangle$$
$$=\left\langle 0,\cos(90^0+45^0),\frac{1}{\sqrt{2}}\right\rangle=\left\langle 0,-\sin 45^0,\frac{1}{\sqrt{2}}\right\rangle$$
$$=\left\langle 0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right\rangle$$

➢ Assertion – Reasons

14. Soln.

(a): If lines are perpendicular, then $\theta=\frac{\pi}{2}$

$$\therefore\ \cos\frac{\pi}{2}=\frac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\Rightarrow\cos\frac{\pi}{2}=\frac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}$$
$$\Rightarrow\ \vec{b}_1.\vec{b}_2=0$$

$\therefore$ Both A and R are true and R is the correct explanation of A.

14. Soln. (d): The given equation of lines can be rewritten as

$$\frac{x-0}{1/2}=\frac{y-0}{1/3}=\frac{z-0}{-1} \text{ and}$$

$$\frac{x-0}{1/6}=\frac{y-0}{-1}=\frac{z-0}{-1/4}$$

$$\therefore\ a_1=\frac{1}{2},\ b_1=\frac{1}{3},\ c_1=-1$$

And $a_2=\frac{1}{6},\ b_2=-1,\ c_2=\frac{-1}{4}$

Now, $\cos\theta=\dfrac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

$$=\frac{\frac{1}{2}\cdot\frac{1}{6}+\frac{1}{3}\cdot(-1)+(-1)\cdot\left(\frac{-1}{4}\right)}{\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{3}\right)^2+(-1)^2}\sqrt{\left(\frac{1}{6}\right)^2+(-1)^2+\left(\frac{-1}{4}\right)^2}}=0$$

$$\Rightarrow\ \cos\theta=0 \Rightarrow \theta=90^0$$

15. Soln. (d): The given lines are perpendicular, if

$a_1a_2+b_1b_2+c_1c_2=0$ ....(i)

Here, $L_1:\dfrac{x-5}{0}=\dfrac{y-0}{3-\alpha}=\dfrac{z-0}{-2}$

$L_2:\dfrac{x-2}{0}=\dfrac{y-0}{-1}=\dfrac{z-0}{2-\alpha}$

Here, a_1,b_1,c_1 are $0,3-\alpha,-2$ and a_2,b_2,c_2 are $0,-1,2-\alpha$ respectively.

$\therefore\ 0\times0-(3-\alpha)-2(2-\alpha)=0$

$\Rightarrow\ \alpha=\dfrac{7}{3}$ [From (i)]

➢ Case Study Questions

16. **Sol.** (i) Clearly, the coordinates of A are (8, 10, 0) and D are (0, 0, 30)

∴ Equation of AD is given by

$$\frac{x-0}{8-0}=\frac{y-0}{10-0}=\frac{z-30}{-30}$$

$$\Rightarrow \frac{x}{4}=\frac{y}{5}=\frac{30-z}{15}$$

(ii) The coordinates of point C are (15, - 20, 0) and D are (0, 0, 30)

∴ Length of the cable DC

$=\sqrt{(0-15)^2+(0+20)^2+(30-0)^2}$

$=\sqrt{225+400+900}=\sqrt{1525}=5\sqrt{61}$ m

(iii) Since, the coordinates of point B are (-6, 4, 0) and D are (0, 0, 30), therefore vector DB is

$(-6-0)\hat{i}+(4-0)\hat{j}+(0-30)\hat{k}$, i.e., $-6\hat{i}+4\hat{j}-30\hat{k}$

(iv) Required sum

$=(8\hat{i}+10\hat{j}-30\hat{k})+(-6\hat{i}+4\hat{j}-30\hat{k})+(15\hat{i}-20\hat{j}-30\hat{k})$

$=17\hat{i}-6\hat{j}-90\hat{k}$

17. Sol. (i) Equation of line joining B and C is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

$$\frac{x-1}{4}=\frac{y-4}{0}=\frac{z-6}{-2}=\lambda$$

Let coordinates of foot of perpendicular be $D(4\lambda+1, 4, -2\lambda+6)$ ... (i)

∴ D.R.'s of AD are $(4\lambda, 2, -2\lambda+5)$.

Now, $4(4\lambda)+0(2)+(-2)(-2\lambda+5)=0$

$\lambda=\frac{1}{2}$

Putting in eqn (i)

Coordinates of D are (3, 4, 5)

∴ Required coordinates are (3, 4, 5).

(ii) Let a, b, c be the direction ratios of the required line. Since it is perpendicular to the lines whose direction ratios are (1,-2, -2) and (0, 2, 1) respectively.

∴ a - 2b - 2c = 0 ...(i)

a + 2b + c = 0 ...(ii)

On solving (i) and (ii) by cross-multiplication, we get

$\frac{a}{-2+4}=\frac{b}{0-1}=\frac{c}{2} \Rightarrow \frac{a}{2}=\frac{b}{-1}=\frac{c}{2}$

Thus, the direction ratios of the required line are (2, -1, 2)

(iii) Direction ratio of given lines are (3, -2, 0) and (1, $\frac{3}{2}$, 2)

Now,

as $3\cdot1+(-2)\cdot\left(\frac{3}{2}\right)+0\cdot2=3-3+0=0$

∴ Given lines are perpendicular to each other.

(iv) Since, direction ratio's are proportional to direction cosine's,

therefore L_1 will be parallel to L_2, iff

$$\frac{l_1}{l_2}=\frac{m_1}{m_2}=\frac{n_1}{n_2}$$

➢ Questions

18. Soln. The given line is 5x – 3 = 15y + 7 = 3 – 10z

$$\Rightarrow \frac{x-\frac{3}{5}}{\frac{1}{5}} = \frac{y+\frac{7}{15}}{\frac{1}{15}} = \frac{z-\frac{3}{10}}{-\frac{1}{10}}$$

Its direction ratios are $\frac{1}{5}, \frac{1}{15}, -\frac{1}{10}$

i.e., Its direction ratios are proportional to 6, 2, -3.

Now, $\sqrt{6^2+2^2+(-3)^2} = 7$

$\therefore$ Its direction cosines are $\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}$.

19. Soln. The cartesian equation of line AB is

$$\frac{2x-1}{12} = \frac{y+2}{2} = \frac{z-3}{3}$$

Can be rearranged as $\frac{x-\frac{1}{2}}{6} = \frac{y+2}{2} = \frac{z-3}{3}$

So, a = 6, b = 2, c = 3

$$\Rightarrow \sqrt{a^2+b^2+c^2} = \sqrt{6^2+2^2+3^2} = 7$$

$\therefore$ Required direction cosines are $\frac{6}{7}, \frac{2}{7}, \frac{3}{7}$.

20. Soln. Any point on the line

$$\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7} = r \text{ (say)} \qquad \text{........(i)}$$

Is (3r – 1, 5r – 3, 7r – 5).

Any point on the line

$$\frac{x-2}{11} = \frac{y-4}{3} = \frac{z-6}{5} = k \text{ (say)} \qquad \text{.........(ii)}$$

Is $(k+2, 3k+4, 5k+6)$

For lines (i) and (ii) to intersect, we must have

$$3r-1 = k+2, 5r-3 = 3k+4, 7r-5 = 5k+6$$

On solving these, we get $r = \frac{1}{2}, k = -\frac{3}{2}$

$\therefore$ Lines (i) and (ii) intersect and their point of intersection is $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$

21. Soln. Let the direction ratios of required line be a, b, c, since, the line is perpendicular to

$$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$$

And $\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$

$\therefore \quad 3a - 16b + 7c = 0$

and $\quad 3a + 8b - 5c = 0.$

Solving by cross multiplication, we get

$$\frac{a}{80-56} = \frac{b}{21+15} = \frac{c}{24+48}$$

$$\Rightarrow \frac{a}{24} = \frac{b}{36} = \frac{c}{72}$$

$\therefore$ the direction ratios of line : 2,3,6.

Hence, required line through the point (1, 2, -4) is

$$\vec{r} = \vec{a} + \lambda\vec{b}$$

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

22. Soln. The equation of a plane passing through (-1,2,0) is

$$a(x+1) + b(y-2) + c(z-0) = 0$$

..........(i)

It passes through (2, 2, -1)

$\therefore \quad a(2+1) + b(2-2) + c(-1-0) = 0$

$a + 0b - c = 0$

..........(ii)

The given lines is

$$\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$$

i.e., $\frac{x-1}{1} = \frac{y+\frac{1}{2}}{1} = \frac{z+1}{-1}$

$\therefore$ d. r'. s of line are 1, 1, -1

The plane (i) is parallel to the given line

$$a + b - c = 0$$

.........................(iii)

Solving (ii) and (iii) by cross multiplication, we get

$$\frac{a}{1} = \frac{b}{2} = \frac{c}{3}$$

i.e., direction ratios of normal to the plane are 1, 2, 3.

$\therefore \quad 1(x+1) + 2(y-2) + 3(z-0) = 0$

i.e., $x + 2y + 3z = 3$

Ans.

23. Soln. The given lines are

$$l_1 : \frac{1-x}{3} = \frac{7y-14}{p} = \frac{z-3}{2}$$

$$\Rightarrow \quad l_1 : \frac{x-1}{-3} = \frac{y-2}{p/7} = \frac{z-3}{2} \quad .....(i)$$

And $l_2 : \dfrac{7-7x}{3p} = \dfrac{y-5}{1} = \dfrac{6-z}{5}$

$$\Rightarrow \quad l_2 : \frac{x-1}{-3p/7} = \frac{y-5}{1} = \frac{z-6}{-5}$$

..........(ii)

Since l_1 and l_2 are perpendicular

$$\therefore \quad (-3).\left(-\frac{3p}{7}\right)+\left(\frac{p}{7}\right).1+2.(-5)=0$$

$$\Rightarrow \quad \frac{10p}{7}-10=0$$

$$\Rightarrow \quad p=7$$

$\therefore$ Equation of the line passes through (3, 2, -4) and parallel to l_1 is

$$\frac{x-3}{-3} = \frac{y-2}{1} = \frac{z+4}{2}$$

24. Soln. Getting

$$3\vec{a}+2\vec{a} = 7\hat{i}-5\hat{j}+4\hat{k}$$

$\therefore$ D.R's are 7, -5, 4.

25. Soln. D-Cosines of line are $\dfrac{1}{2}, -\dfrac{1}{2}, \dfrac{1}{\sqrt{2}}$

Equation of line is:

$$\frac{x-2}{1/2} = \frac{y+3}{\frac{-1}{2}} = \frac{z-4}{\frac{1}{\sqrt{2}}}$$

Or $\quad 2x-4=-2y-6$

$$=\sqrt{2}(z-4)$$

26. Soln. Vector form of a line is given as

$$\vec{r} = 2\hat{i}+\hat{j}-4\hat{k}+\lambda(\hat{i}-\hat{j}-\hat{k})$$

Direction ratios of above equation are (1, -1, -1) and point through which the line passes is (2, 1, -4).

$\therefore$ Cartesian equation is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

$$= \frac{x-2}{1} = \frac{y-1}{-1} = \frac{z+4}{-1} \quad or \quad x-2=1-y=-z-4$$

27. Soln. The equation of such plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

28. Soln. Here, D.R's or required line are (1, 2, -2)

Passing through (1, -1, 2)

Therefore, vector equation of line will be

$$\vec{r}(\hat{i}-\hat{j}+2\hat{k})+\lambda(\hat{i}+2\hat{j}-2\hat{k})$$

Or $\quad \dfrac{x-1}{1} = \dfrac{y+1}{2} = \dfrac{z-2}{-2}$

29. Soln. Given,

$$\frac{3-x}{-3} = \frac{y+2}{-2} = \frac{z+2}{6}$$

Or $\quad \dfrac{x-3}{3} = \dfrac{y+2}{-2} = \dfrac{z+2}{6}$

Direction ratios are 3, -2, 6.

DRF Parallel line 3, -2, 6

D.C's [illegible] n

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}},$$

$$h = \frac{b}{\sqrt{a^2+b^2+c^2}},$$

$$n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

$$D.C's = \frac{3}{\sqrt{9+4+36}}, \frac{-2}{\sqrt{9+4+36}}, \frac{6}{\sqrt{9+4+36}}$$

$$D.C's\ are\ \frac{3}{7}, \frac{-2}{7}, \frac{6}{7}$$

30. Soln. $\vec{r_1} = \hat{i}+2\hat{j}-\hat{k}+\lambda(2\hat{i}-3\hat{j}+4\hat{k})$

And $\quad \vec{r_2} = -2\hat{i}+3\hat{j}+\mu(-\hat{i}+2\hat{j}+3\hat{k})$

Let $\quad \vec{a_1} = \hat{i}+2\hat{j}-\hat{k},\ \vec{b_1} = 2\hat{i}-3\hat{j}+4\hat{k}$

And $\quad \vec{a_2} = -2\hat{i}+3\hat{j},\ \vec{b_2} = -\hat{i}+2\hat{j}+3\hat{k}$

$$\vec{a_2}-\vec{a_1} = -3\hat{i}+\hat{j}+\hat{k}$$

$$\vec{b_1}\times\vec{b_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -3 & 4\\ -1 & 2 & 3\end{vmatrix}$$

$$=-17\hat{i}-10\hat{j}+\hat{k}$$

The required shortest distance

$$=\frac{|(\vec{a_2}-\vec{a_1}).(\vec{b_1}\times\vec{b_2})|}{|\vec{b_1}\times\vec{b_2}|}$$

$$=\frac{(-3\hat{i}+\hat{j}+\hat{k})(-17\hat{i}-10\hat{j}+\hat{k})}{\sqrt{(-17)^2+(-10)^2+(1)^2}}$$

$$=\frac{42}{\sqrt{390}}\quad units$$

31. Soln. General points on the lines are

$$(1+3\lambda)\hat{i}+(1-\lambda)\hat{j}-\hat{k}$$

And $(4+2\mu)\hat{i}+3(3\mu-1)\hat{k}$

Lines intersect if

$$1+3\lambda=4+2\mu \text{.......(i)}$$

$$1-\lambda=0 \qquad \text{.......(ii)}$$

$$and \quad 3\mu-1=-1 \qquad \text{.......(iii)}$$

For some λ & μ

From (ii) & (iii), $\lambda=1,\ \mu=0$

Substituting in equation (i)

Since, $1+3(1)=4+2(0)$ is true

$\therefore$ lines intersect

Point of intersection is:

$4\hat{i}-\hat{k}$ or (4, 0, -1)

32. Soln. The equation of line passing through the points (3, -4, -5) and (2, -3, 1) is

$$\frac{x-3}{2-3}=\frac{y+4}{-3+4}=\frac{z+5}{1+5}\Rightarrow$$

$$\frac{x-3}{-1}=\frac{y+4}{1}=\frac{z+5}{6} \qquad \text{..........(i)}$$

Let the line (i) crosses at point $P(\alpha,\beta,\gamma)$ the plane

$$2x+y+z=7 \qquad \text{.........(ii)}$$

$\because$ P lies on line (i), therefore (α,β,γ) satisfy equation (i)

$$\therefore \quad \frac{\alpha-3}{-1}=\frac{\beta+4}{1}=\frac{\gamma+5}{6}=\lambda(\text{say})$$

$$\Rightarrow \quad \alpha=-\lambda+3;\ \beta=\lambda-4\ and\ \gamma=6\lambda-5$$

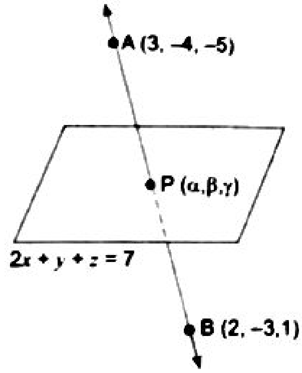
Also $P(\alpha,\beta,\gamma)$ lie on plane (ii)

$$\therefore \quad 2\alpha+\beta+\gamma=7$$

$$\Rightarrow \quad 2(-\lambda+3)+(\lambda-4)+(6\lambda-5)=7$$

$$\Rightarrow \quad -2\lambda+6+\lambda-4+6\lambda-5=7$$

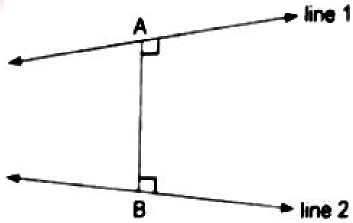
$$\Rightarrow \quad 5\lambda=10\Rightarrow\lambda=2$$



Hence, the coordinate of required point P is $(-2+3,2-4,6\times2-5)$ i.e., $(1,-2,7)$

33. Soln. Let $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}=\lambda$ and $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}=k$

Now let us take a point on first line as $A(\lambda+3,-2\lambda+5,\lambda+7)$ and let $B(7k-1,-6k-1,k-1)$ be point on the second line



The direction ratio of the line AB

$$7k-\lambda-4,\ -6k+2\lambda-6,k-\lambda-8$$

Now, as AB is the shortest distance between line 1 and line 2 so,

$$(7k-\lambda-4)\times1+(-6k+2\lambda-6)\times(-2)+(k-\lambda-8)\times1=0$$

..........(i)

And

$$(7k-\lambda-4)\times7+(-6k+2\lambda-6)\times(-6)+(k-\lambda-8)\times1=0$$

.........(ii)

Solving equation (i) and (ii), we get

$$\lambda=0\ and\ k=0$$

$$\therefore \quad A\equiv(3,5,7)\text{ and }B\equiv(-1,-1,-1)$$

$\therefore \quad AB = \sqrt{(3+1)^2 + (5+1)^2 + (7+1)^2} = \sqrt{16+36+64}$

$= \sqrt{116}$ units = $2\sqrt{29}$ units

34. Soln. Let the cartesian equation of line passing through (1, 2, -4) be

$$\frac{x-1}{a} = \frac{y-2}{b} = \frac{z+4}{c}$$

.............(i)

Given lines are

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$$

........(ii)

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$$

.........(iii)

Obviously parallel vectors $\vec{b_1}, \vec{b_2}$ and $\vec{b_3}$ of (i), (ii) and (iii) respectively are given as

$\vec{b_1} = a\hat{i} + b\hat{j} + c\hat{k}; \ \vec{b_2} = 3\hat{i} - 16\hat{j} + 7\hat{k}; \ \vec{b_3} = 3\hat{i} + 8\hat{j} - 5\hat{k}$

According to question

(i) $\perp$ (ii)

$\Rightarrow \quad \vec{b_1} \perp \vec{b_2} \quad \Rightarrow \quad \vec{b_1} \cdot \vec{b_2} = 0$

(i) $\perp$ (iii) $\Rightarrow \quad \vec{b_1} \perp \vec{b_3} \quad \Rightarrow \quad \vec{b_1} \cdot \vec{b_3} = 0$

Hence, 3a – 16b + 7c = 0

.........(iv)

And 3a + 8b – 5c = 0

.........(v)

From equation (iv) and (v), we get

$$\frac{a}{80-56} = \frac{b}{21+15} = \frac{c}{24+48}$$

$\Rightarrow \quad \frac{a}{24} = \frac{b}{36} = \frac{c}{72} \quad \Rightarrow \quad \frac{a}{2} = \frac{b}{3} = \frac{c}{6} = \lambda$ (say)

$\Rightarrow \quad a = 2\lambda, \ b = 3\lambda, \ c = 6\lambda$

Putting the value of a, b, c in (i), we get the required cartesian equation of line as

$$\frac{x-1}{2\lambda} = \frac{y-2}{3\lambda} = \frac{z+4}{6\lambda} \quad \Rightarrow \quad \frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

Hence, vector equation is

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

35. Soln. Given line is

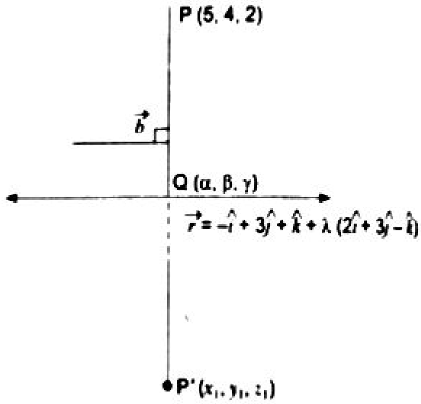
$$\vec{r} = -\hat{i} + 3\hat{j} + \hat{k} + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$$

It can be written in cartesian form as

$$\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1}$$

...........(i)

Let Q (α, β, γ) be the foot of perpendicular drawn from



P(5, 4, 2) to the line (i) and P' (x_1, y_1, z_1) be the image of P on the line (i)

$\because \quad Q(\alpha, \beta, \gamma)$ lie on line (i)

$\therefore \quad \frac{\alpha+1}{2} = \frac{\beta-3}{3} = \frac{\gamma-1}{-1} = \lambda$ (say)

$\Rightarrow \quad \alpha = 2\lambda - 1; \ \beta = 3\lambda + 3$ and $\gamma = -\lambda + 1$

..........(ii)

Now, $\overrightarrow{PQ} = (\alpha-5)\hat{i} + (\beta-4)\hat{j} + (\gamma-2)\hat{k}$

Parallel vector of line (i) $\vec{b} = 2\hat{i} + 3\hat{j} - \hat{k}$.

Obviously $\overrightarrow{PQ} \perp \vec{b} \quad \Rightarrow \quad \overrightarrow{PQ} \cdot \vec{b} = 0$

$2(\alpha+5) + 3(\beta-4) + (-1)(\gamma-2) = 0$

$\Rightarrow \quad 2\alpha - 10 + 3\beta - 12 - \gamma + 2 = 0$

$\Rightarrow \quad 2\alpha + 2\beta - \gamma - 20 = 0$

$\Rightarrow \quad 2(2\lambda-1) + 3(3\lambda+3) - (-\lambda+1) - 20 = 0$ [Putting value of α, β, γ from (ii)]

$\Rightarrow \quad 4\lambda - 2 + 9\lambda + 9 + \lambda - 1 - 20 = 0$

$\Rightarrow \quad 14\lambda - 14 = 0 \quad \Rightarrow \quad \lambda = 1$

Hence the coordinates of foot of perpendicular Q are (2 x 1 - 1, 3 x 1 + 3, -1 + 1) i.e., (1, 6, 0)

$\therefore$ Length of perpendicular =

$\sqrt{(5-1)^2 + (4-6)^2 + (2-0)^2}$

$= \sqrt{16+4+4} = \sqrt{24} = 2\sqrt{6}$

units.

Also, since Q is mid – point of PP'

$$\therefore \quad 1=\frac{x_1+5}{2} \quad \Rightarrow \quad x_1=-3$$

$$6=\frac{y_1+4}{2} \quad \Rightarrow \quad y_1=8$$

$$0=\frac{z_1+2}{2} \quad \Rightarrow \quad z_1=-2$$

Therefore required image is (-3, 8, -2).

36. Soln. The equation of line AB is given by

$$\frac{x-0}{4-0}=\frac{y+1}{5+1}=\frac{z+1}{1+1}=\lambda \text{ (say)}$$

$$\Rightarrow \quad x=4\lambda,\ y=6\lambda-1,\ z=2\lambda-1$$

The coordinates of a general point on AB are

$(4\lambda, 6\lambda, -1, 2\lambda-1)$

The equation of line CD is given by

$$\frac{x-3}{3+4}=\frac{y-9}{9-4}=\frac{z-4}{4-4}=\mu \text{ (say)}$$

$$\Rightarrow \quad x=7\mu+3,\ y=5\mu+9,\ z=4$$

The coordinates of a general point on CD are

$(7\mu+3, 5\mu+9, 4)$

If the line AB and CD intersect then they have a common point. So, for some values of λ and μ, we must have

$$4\lambda=7\mu+3,\ 6\lambda-1=5\mu+9,\ 2\lambda-1=4$$

$$\Rightarrow 4\lambda-7\mu=3 \text{ .....(i)},\ 6\lambda-5\mu=10 \text{.......(iii) and } \lambda=\frac{5}{2} \text{ ....(iii)}$$

Substituting $\lambda=\frac{5}{2}$ *in* (ii), we get $\mu=1$

Since $\lambda=\frac{5}{2}$ and $\mu=1$ satisfy (i), so the given lines AB and CD intersect.

37. Sol. Any point on the line

$$\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}=r \text{ (say)}$$

.........(i)

Is (3r – 1, 5r – 3, 7r – 5).

Any point on the line

$$\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}=k \text{(say)}$$

.......(ii)

Is $(k+2, 3k+4, 5k+6)$

For lines (i) and (ii) to intersect, we must have

$$3r-1=k+2,\ 5r-3=3k+4,\ 7r-5=5k+6$$

On solving these, we get $r=\frac{1}{2},\ k=-\frac{3}{2}$

$\therefore$ Lines (i) and (ii) intersect and their point of intersection is $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$

38. Soln. The given lines are

$\vec{r}=3\hat{i}+2\hat{j}-4\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k})$ and

$\vec{r}=5\hat{i}-2\hat{j}+\mu(3\hat{i}+2\hat{j}+6\hat{k})$

$$\Rightarrow \vec{r}=(3+\lambda)\hat{i}+(2+2\lambda)\hat{j}+(2\lambda-4)\hat{k} \qquad .....(i)$$

$$and\ \vec{r}=(5+3\mu)\hat{i}+(2\mu-2)\hat{j}+6\mu\hat{k} \qquad ....(ii)$$

If these lines intersect, they must have a common point.

So, we must have

$$(3+\lambda)\hat{i}+(2+2\lambda)\hat{j}+(2\lambda-4)\hat{k}=(5+3\mu)\hat{i}+(2\mu-2)\hat{j}+6\mu\hat{k}$$

$$\Rightarrow \quad 3+\lambda=5+3\mu \quad \Rightarrow \quad \lambda-3\mu=2,$$

$$2+2\lambda=2\mu-2 \quad \Rightarrow \quad \lambda-\mu=-2$$

$$and \quad 2\lambda-4=6\mu \quad \Rightarrow \quad \lambda-3\mu=2$$

$$\Rightarrow \quad \lambda=-4,\ \mu=-2.$$

$\therefore$ The given lines intersect and their point of intersection is (-1, -6, -12).

39. Soln. Let θ be the angle made by the line with y-axis.

Then, $\cos^2 60^0+\cos^2 45^0+\cos^2\theta=1$

$$\Rightarrow \quad \frac{1}{4}+\frac{1}{2}+\cos^2\theta=1$$

$$\Rightarrow \quad \cos^2\theta=1-\frac{1}{2}-\frac{1}{4}\Rightarrow \cos\theta=\frac{1}{2}\Rightarrow \theta=60^0$$

$\therefore$ Direction cosines of the line are

$$<\cos 60^0, \cos 45^0, \cos 60^0> i.e., \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}.$$

40. Soln. The given lines are

$$l_1: \frac{x-1}{-3}=\frac{y-2}{\lambda/7}=\frac{z-3}{2}$$

And $l_2: \dfrac{x-1}{-3\lambda/7}=\dfrac{y-5}{1}=\dfrac{z-6}{-5}$

Now, $l_1 \perp l_2$

[Given]

$$\therefore \quad (-3)\left(-\frac{3\lambda}{7}\right)+\frac{\lambda}{7}-10=0$$

$$\Rightarrow \quad \frac{9\lambda}{7}+\frac{\lambda}{7}-10=0 \Rightarrow \frac{10\lambda}{7}=10 \Rightarrow \lambda=7$$

Since for $\lambda=7$, given lines are at right angle.

$\therefore$ Lines are intersecting.

41. Soln. The given lines are

$$\vec{r}=(8\hat{i}-19\hat{j}+10\hat{k})+\lambda(3\hat{i}-16\hat{j}+7\hat{k})$$

and $\vec{r}=(15\hat{i}+29\hat{j}+5\hat{k})+\mu(3\hat{i}+8\hat{j}-5\hat{k})$

Equation of any line through (1, 2, -4) with d.r's l, m, n is

$$\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+p(l\hat{i}+m\hat{j}+n\hat{k}) \quad ......(i)$$

Since, the required line is perpendicular to both the given lines.

$$\therefore \quad 3l-16m+7n=0 \text{ and } 3l+8m-5n=0$$

$$\Rightarrow \quad \frac{l}{80-56}=\frac{m}{21+15}=\frac{n}{24+48} \Rightarrow \frac{l}{2}=\frac{m}{3}=\frac{n}{6}$$

$\therefore$ From (i), the required line is

$$\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+p(2\hat{i}+3\hat{j}+6\hat{k})$$

Here, the position vector of passing point is

$\vec{a}=\hat{i}+2\hat{j}-4\hat{k}$ and parallel vector is

$\vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$

$\therefore$ Cartesian equation of line is given by

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}$$

42. Soln. The given lines are

$$l_1: \frac{x-1}{-3}=\frac{y-2}{p/7}=\frac{z-3}{2}$$

$$l_2: \frac{x-1}{-3p/7}=\frac{y-5}{1}=\frac{z-6}{-5}$$

$\because$ l_1 is perpendicular to l_2.

$$\therefore \quad (-3)\left(\frac{-3p}{7}\right)+\frac{p}{7}.1+2(-5)=0$$

$$\Rightarrow \quad \frac{9p}{7}+\frac{p}{7}=10 \Rightarrow \frac{10p}{7}=10 \Rightarrow p=7$$

Now, eq. of the line passing through (3,2,-4) and parallel to l_1 is

$$\frac{x-3}{-3}=\frac{y-2}{1}=\frac{z+4}{2}.$$

43. Soln. The given line is

$$\frac{x+2}{2}=\frac{2y-7}{6}=\frac{5-z}{6}$$

$$\Rightarrow \quad \frac{x+2}{2}=\frac{y-\frac{7}{2}}{3}=\frac{z-5}{-6} \quad ....(i)$$

Its d.r's are 2, 3, -6

$$\because \quad \sqrt{2^2+3^2+(-6)^2}=7$$

$$\because \quad Its\ d.c's\ are\ \frac{2}{7},\frac{3}{7},-\frac{6}{7}$$

Eq. of a line through (-1, 2, 3) and parallel to (i) is

$$\frac{x+1}{2}=\frac{y-2}{3}=\frac{z-3}{-6}=\lambda \text{ (say)}$$

$\therefore$ Vector equation of a line passing through (-1, 2, 3) and parallel to (i) is given by

$$\vec{r}=(-\hat{i}+2\hat{j}+3\hat{k})+\lambda(2\hat{i}+3\hat{j}-6\hat{k}).$$

44. Soln. The given lines are

$\vec{r}=(\hat{i}+2\hat{j}+\hat{k})+\lambda(\hat{i}-\hat{j}+\hat{k})$ and

$$\vec{r}=(2\hat{i}-\hat{j}-\hat{k})+\mu(2\hat{i}+\hat{j}+2\hat{k})$$

On comparing, we get

$$\vec{a_1}=\hat{i}+2\hat{j}+\hat{k},\ \vec{b_1}=\hat{i}-\hat{j}+\hat{k}$$

$$\vec{a_2}=2\hat{i}-\hat{j}-\hat{k},\ \vec{b_2}=2\hat{i}+\hat{j}+2\hat{k}$$

$$\therefore \quad \vec{a_2}-\vec{a_1}=\hat{i}-3\hat{j}-2\hat{k}$$

$$\vec{b_1}\times\vec{b_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 1\\ 2 & 1 & 2\end{vmatrix}=-3\hat{i}+0.\hat{j}+3\hat{k}$$

$$\Rightarrow \quad |\vec{b_1}\times\vec{b_2}|=\sqrt{(-3)^2+3^2}=3\sqrt{2}$$

$$\Rightarrow \quad (\vec{a_2}-\vec{a_1}).(\vec{b_1}\times\vec{b_2})=1(-3)-3(0)-2(3)=-9.$$

$$\therefore \quad d=\left|\frac{-9}{3\sqrt{2}}\right|=\frac{3}{\sqrt{2}}=\frac{3}{2}\sqrt{2}\ units$$

SURE SHOT QUESTIONS

Chapter – 12

Linear Programming

➢ MCQ (1 mark)

1. Soln. (b): Construct the following table of values of the objective function:

Corner Point	Value of Z = 4x + 3y
(0,0)	$4\times0+3\times0=0$
(0, 40)	$4\times0+3\times40=120$
(20, 40)	$4\times20+3\times40=200$
(60, 20)	$4\times60+3\times20=300$
(60, 0)	$4\times60+3\times0=240$

2. Soln. (b): Construct the following table of values of the objective function:

Corner Points	Value of Z = 3x – 4y
(0, 0)	$3\times0-4\times0=0$
(5, 0)	$3\times5-4\times0=15$
(6, 5)	$3\times6-4\times5=-2$
(6, 8)	$3\times6-4\times8=-14$
(4, 10)	$3\times4-4\times10=-28$
(0, 8)	$3\times0-4\times8=-32$

Minimum of Z = -32 at (0, 8)

3. Soln. (a): Construct the following table of values of the objective function F:

Corner Point	Value of $F = 3x - 4y$	
(0, 0)	$3\times0-4\times0=0$	← Maximum
(6, 12)	$3\times6-4\times12=-30$	
(6, 16)	$3\times6-4\times16=-46$	← Minimum
(0, 4)	$3\times0-4\times4=-16$	

Hence, maximum of F = 0

4. Soln. (d): Construct the following table of values of objective function:

Corner Point	Value of $F = 4x + 6y$	
(0, 2)	$4\times0+6\times2=12$	} ← Minimum
(3, 0)	$4\times3+6\times0=12$	
(6, 0)	$4\times6+6\times0=24$	
(6, 8)	$4\times6+6\times8=72$	← Maximum
(0, 5)	$4\times0+6\times5=30$	

Since the minimum value (F) = 12 occurs at two distinct corner points, it occurs at every point of the segment joining these two points.

5. Soln. (b): We must have value of Z at (3, 0) = value of Z at (1, 1)

$$\Rightarrow 3p+0.q=1.p+1.q\Rightarrow 3p=p+q\Rightarrow p=\frac{1}{2}q$$

6. Soln.

(d): We have, $2x+y\le10$ and $x+2y\ge8$

Let us check which of the given points satisfy the given inequation one by one.

(a) (-2, 4)

$2\times(-2)+4=-4+4=0\le10$

And $-2+2\times4=-2+8=6\ge8$

(b) (3, 2)

$2\times3+2=6+2=8\le10$

$3+2\times2=3+4=7\ge8$

(c) (-5, 6)

$2\times(-5)+6=-10+6=-4\le10$

$-5+2\times6=-5+12=7\ge8$

(d) (4, 2)

$2\times4+2=10\le10; 4+2\times2=8\ge8$

$\therefore$ (4,2) satisfy both the inequations.

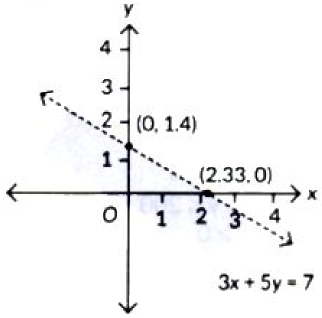
7. Soln.

(c): Given inequation is 3x + 5y < 7

Let us draw the graph of 3x + 5y = 7

X	0	2.33
Y	1.4	0

Substitute, x = 0 and y = 0 in the inequation, we get



3(0) + 5(0) < 7

i.e., 0 < 7 which is true.

∴ The solution set of the inequality is an open half plane containing the origin except the points on line 3x + 5y = 7.

8. Soln.

(a): Given, Z = 11x + 7y

At (0, 3), Z = 11 x 0 + 7 x 3 = 21

At (3, 2), Z = 11 x 3 + 7 x 2 = 47

At (0, 5), Z = 11 x 0 + 7 x 5 = 35

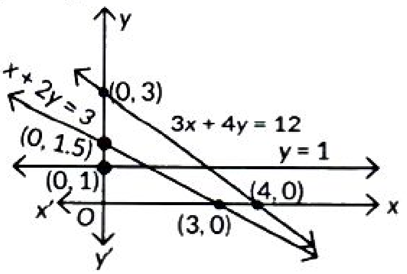
Thus, Z is minimum at (0, 3) and minimum value of Z is 21.

9. Soln.

(a): Given,

$x + 2y \leq 3,\ 3x + 4y \geq 12, x \geq 0, y \geq 1$

The graph of given constraints is shown here.

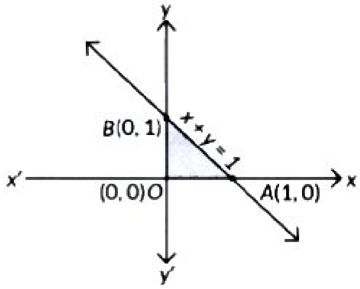


Since, there is no common region, so no solution exists.

10. Soln.

(b): We have to maximise Z = 3x + 4y

Subject to constraints, $x \geq 0,\ y \geq 0$ and $x + y \geq 1$



The shaded portion OAB is the feasible region, where

O(0, 0), A(1, 0) and B(0, 1) are the corner points.

At O(0, 0), Z = 3 x 0 + 4 x O = 0

At A(1, 0), Z = 3 x 1 + 4 x 0 = 3

At B(0, 1), Z = 3 x 0 + 4 x 1 = 4

∴ Maximum value of Z is 4, which occurs at B(0, 1).

11. Soln. (b): Clearly, the pair of points given in graph, and (0, 104); (52, 0) and (0, 38); (76, 0) satisfy the corresponding linear equations given in option(b) i.e., 2x + y ≤ 104 and $x + 2y \leq 76$.

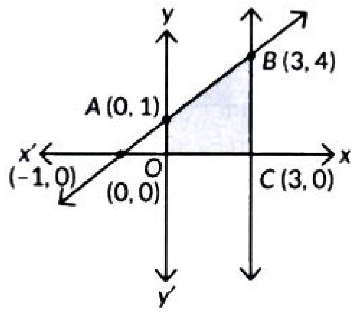
12. Soln. (b): Since, minimum value of Z = ax + by occurs at two points (3, 4) and (4, 3).

$\therefore\ 3a + 4b = 4a + 3b \Rightarrow a = b$

13. Soln. (c): Given, Z = 3x + 4y

Subject to constraints,

$x - y \geq -1,\ x \leq 3;\ x \geq 0, y \geq 0$



The shaded region OABC is the feasible region, where corner points are O(0, 0), A(0, 1), B(3, 4) and C(3, 0)

At O(0, 0), Z = 3(0) + 4(0) = 0

At A(0, 1), Z = 3(0) + 4(1) = 4
At B (3, 4), Z = 3(3)+4(4) = 25
At C (3, 0), Z = 3(3) + 4(0) = 9
∴ Maximum value of Z is 25, which occurs at B(3, 4).

14. Soln. (a): Since, maximum value of z = ax + by occurs at both (2, 4) and (4, 0).

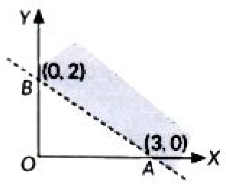
$$\therefore\ 2a+4b=4a+0 \Rightarrow 4b=2a \Rightarrow 2b=a$$

15. Soln. (d): In an LPP, if the objective function z = ax + by has the same maximum value on two corner points of the feasible region, then the number of points at which z_{max} occurs is infinite.

16. Soln. (b): We know that minimum of objective function occurs at corner points.

Corner points	Value of z = 3x – 4y
(0, 0)	0
(5, 0)	15
(6, 5)	[illegible]
(6, 8)	-1[illegible]
(4, 10)	-28
(0, 8)	-32 ⟵ Minimum

17. Soln. (b): From the graph of inequality 2x + 3y > 6. It is clear that it does not contain the origin nor the points of the line 2x + 3y = 6.



18. Soln. (b): A linear function to be optimized is called an objective function.

19. Soln. (b): The strict inequality represents an open half plane and it contains the origin, as (0, 0) satisfies it.

20. Soln. (d): The minimum value of the objective function occurs at two adjacent corner points (0.6, 1.6) and (3, 0) and there is no point in the half plane 4x + 6y < 12 in common with the feasible region.

So, the minimum value occurs at every point of the line segment joining the two points.

21. Soln. (d): We have,

Corner points	Value of Z = 3x + 9y
A(0, 10)	3 x 0 + 9 x 10 = 90
B (5, 5)	3 x 5 + 9 x 5 = 60
C(15, 15)	3 x 15 + 9 x 15 = 180 (Maximum)
D(0, 20)	3 x 0 + 9 x 20 = 180 (Maximum)

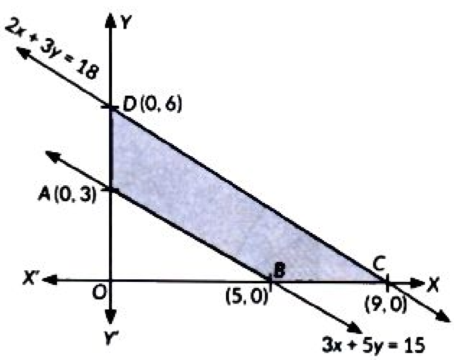
∵ Z is maximum at C(15, 15) and D(0, 20).
∴ Z is maximum at every point on the line joining CD.

22. Soln. ([illegible]): We have,

Corner points	Value of Z = 2x – 3y
(0, 0)	2 × 0 – 3 × 0 = 0
(0, 8)	2 × 0 – 3 × 8 = –24 (Minimum)
(4, 10)	2 × 4 – 3 × 10 = –22
(6, 8)	2 × 6 – 3 × 8 = –12
(6, 5)	2 × 6 – 3 × 5 = –3
(5, 0)	2 × 5 – 3 × 0 = 10

∴ Value of Z is minimum at (0, 8).

23. Soln. (c): Here, the feasible region is shaded.



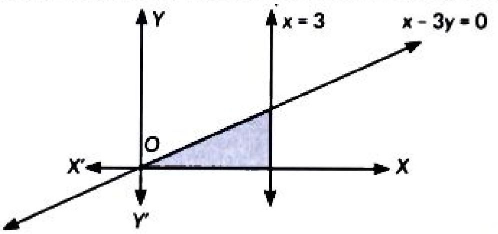
Corner points	Value of $Z = 30x + 50y$
A(0, 3)	$30 \times 0 + 50 \times 3 = 150$ (Minimum)
B(5, 0)	$30 \times 5 + 50 \times 0 = 150$ (Minimum)
C(9, 0)	$30 \times 9 + 50 \times 0 = 270$
D(0, 6)	$30 \times 0 + 50 \times 6 = 300$

Since, minimum value of Z occurs at both A and B. So, Z is minimum at every point on the line joining AB. So, minimum value of Z occurs at infinitely many points.

24. Soln. (a): As, Z is maximum at (30, 30) and (0, 40).

$$\Rightarrow 30a + 30b = 40b \Rightarrow b - 3a = 0$$

25. Soln. (b): From the graph, we can say that the feasible region is bounded in the first quadrant.



➤ Assertion-Reasoning (1 mark)

26. Sol. **(b)** Both A and R are true but R is not the correct explanation of A.

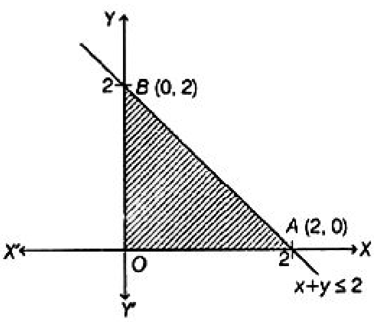
Explanation: Assertion: Given, x + y 2, x 0 and y 0

Let Z = 3x + 2y

Now, table for x + y = 2

x	0	2	1
y	2	0	1

At (0, 0), 0 + 0 2 => 0 2, which is true.



So, shaded portion is towards the origin.

∴The corner points of shaded region are O(0, 0), A(2, 0) and B(0, 2)

At point O(0, 0), Z = 3(0) + 2(0) = 0

At point A(2, 0), Z = 3(2) + 2(0) = 6

At point B(0, 2), Z = 3(0) + 2(2) = 4

Hence, maximum value of Z is 6 at point (2, 0).

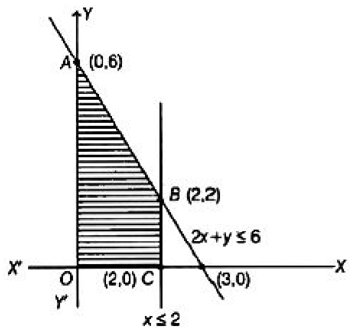
Hence both Assertion and Reason are true but Reason is not the correct explanation of Assertion.

27. Sol. **(b)** Both A and R are true but R is not the correct explanation of A.

Explanation: Both A and R are true but R is not the correct explanation of A.

28. Sol. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation: Assertion: The corresponding graph of the given LPP is



From the above graph, we see that the shaded region is the feasible region OABC which is bounded.

∴The maximum value of the objective function Z occurs at the corner points. The corner points are O(0, 0), A(0, 6), B(2, 2),C(2, 0).

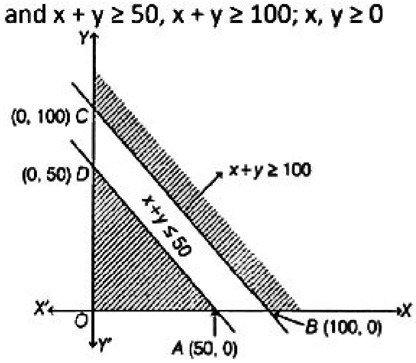
The values of Z at these comer points are given by

Corner point	Corresponding value of Z=11x+7y
(0,0)	0
(0,6)	42←Maximum
(2, 2)	36
(2, 0)	22

Thus, the maximum value of Z is 42 which occurs at the point (0, 6).

29. Sol. **(d)** A is false but R is true.

Explanation: Assertion: Given, maximise, Z = 4x + y and x + y ≥ 50, x + y ≥ 100; x, y ≥ 0



Hence, it is clear from the graph that it is not bounded region. So, maximum value cannot be determined.

Hence Assertion is not true but Reason is true.

30. Sol. **(b)** Both A and R are true but R is not the correct explanation of A.

Explanation: Both A and R are true but R is not the correct explanation of A.

➢ Case Study Questions

31. Ans.

(i)

Corner points	Value of Z=4x-6y
(0,3)	4X0-6X3=-18
(5,0)	4X4-5X0=20
(6, 8)	4X6-6X8=-24
(0, 8)	4X0-6X8=-48

Minimum value of Z is - 48 which occurs at (0, 8).

(ii)

Corner points	Value of Z=4x-6y
(0,3)	4X0-6X3=-18
(5,0)	4X4-5X0=20
(6, 8)	4X6-6X8=-24
(0, 8)	4X0-6X8=-48

Maximum value of Z is 20, which occurs at (5, 0).

(iii)

Corner points	Value of Z=4x-6y
(0,3)	4X0-6X3=-18
(5,0)	4X4-5X0=20
(6, 8)	4X6-6X8=-24
(0, 8)	4X0-6X8=-48

Maximum of Z - Minimum of Z = 20 - (-48) = 20 + 48 = 68

(iv) The corner points of the feasible region are O(0, 0), A(3, 0), B(3, 2), C(2, 3), D(0, 3).

32. Sol. (i) Let number of pairs of earing = x and number of Necklaces = y

As per the given information

x, y≥ 0

0.5x + y≤ 10

x + y≤ 15

Profit function = Z = 30x + 40y

(ii) Let number of pairs of earing = x and number of Necklaces = y

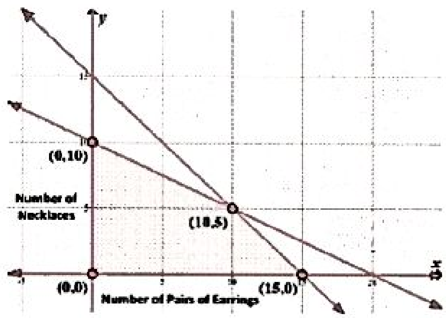
As per the given information

x, y 0

0.5x + y 10

x + y 15

Profit function = Z = 30x + 40y



(iii) From graph corner points are (0, 0), (0, 10), (10, 5) and (15, 0).

Corner points	Maximum profit = Z=30x+40y
(0, 0)	Z=0
(0, 10)	Z=Rs400
(10, 5)	Z=Rs500
(15,0)	Z=Rs 450

Hence profit is maximum when x = number of pair of Earings = 10 and y = Number of Neckleses

(iv) When x = 5 and y = 5

Z = 30x + 40y = 150 + 200 = ₹350

➤ Questions

33. Soln.

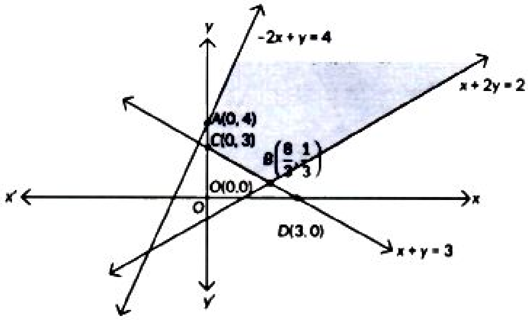
We have, maximise z = -3x – 5y

Converting the given inequations into equations, we get

$-2x+y=4$ ....(i)

$x+y=3$ ....(ii)

$x-2y=2$ ....(iii)



We draw the graph of these lines.

As, $x \geq 0,\ y \geq 0$ so the solution lies in first quadrant.

From graph, corner point of feasible region are A(0, 4), B(8/3, 1/3) and C(0, 3).

The value of z at these corner points are shown as:

Corner points	$z=-3x-5y$
A(0, 4)	-20
B(8/3, 1/3)	-29/3
C(0, 3)	-15

Hence maximum value of $z=\dfrac{-29}{3}$.

34. Soln. We have, maximize $P=70x+40y$

Subject to : $3x+2y \leq 9$

$3x+y \leq 9$

$x \geq 0, y \geq 0$

Convert all inequations into equation, we get

$3x+2y=9$ ......(i)

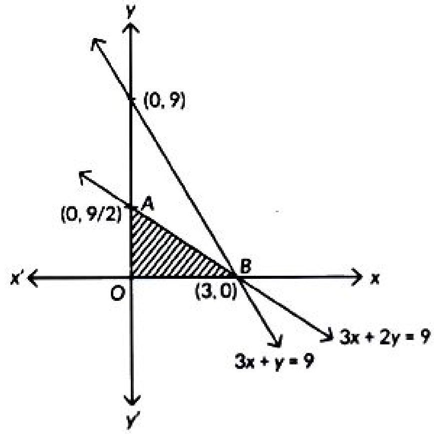
$3x+y=9$ ....(ii)

$x=0$ *and* $y=0$

Solving (i) and (ii), we get

$x=3,\ y=0$

So, point of intersection of equation (i) and (ii) are (3, 0).



The given shaded region is the feasible region.

The corner points of the feasible region are O(0, 0), A(0, 9/2) and B(3, 0).

Corner points	Value of p = 70x + 40y
O(0, 0)	70 x 0 + 40 x 0 = 0
A(0, 9/2)	$70\times0+40\times\dfrac{9}{2}=180$
B(3, 0)	$70\times3+40\times0=210$ (maximum)

So, P is maximum at point B(3, 0).

35. Soln. We first convert the inequalities into equations to obtain lines

$2x+4y=8$ ....(i)

$3x+y=6$ ......(ii)

$x+y=4$ ....(iii)

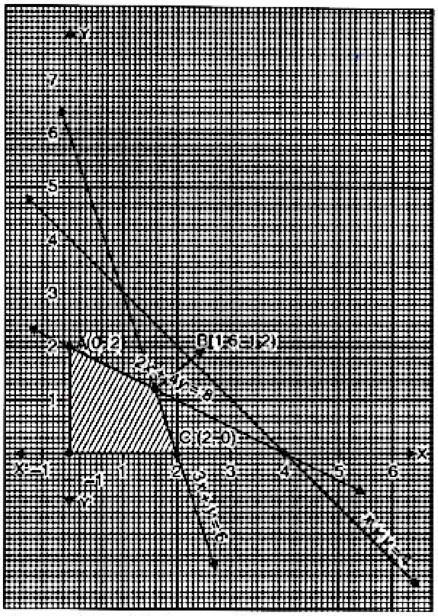
$x=0$

And $y=0$.

We need to maximize the objective function

$z=2x+5y$

These lines are drawn and the feasible region of the L.P.P. is the shaded region:

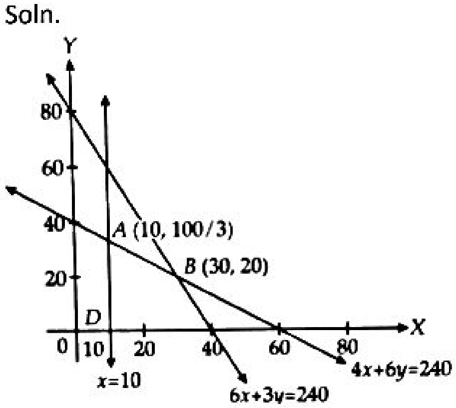


The point of intersection of (i) and (ii) is B (1.6, 1.2)
The coordinates of the corner points of the feasible region are O(0, 0), A(0, 2), B(1.6, 1.2) and C(2, 0).
The value of the objective function at these points are given in the following table:

Corner Points	Value of the objective function $z = 2x + 5y$
O(0, 0)	2 x 0 + 5 x 0 = 0
A (0, 2)	2 x 0 + 5 x 2 = 10 maximum
B(1.6, 1.2)	2 x 1.6 + 5 x 1.2 = 9.2
C(2, 0)	2 x 2 + 5 x 0 = 4

Out of these values of z, the maximum value of z is 10 which is attained at the point (0, 2). Thus the maximum value of z is 10. Ans.

36. Soln.



Maximise z = 7x + 10y, subject to 4x + 6y ≤ 240 ;
$6x+3y \le 240;\ \ x \ge 10,\ x \ge 0,\ y \ge 0$
Correct graph of three lines
For correct shading

$$Z(\mathrm{A}) = Z\left(10, \frac{200}{6}\right) = 70 + 10 \times \frac{100}{3} = 403\frac{1}{3}$$

$$Z(B) = Z(30, 20) = 210 + 200 = 410$$

$$Z(\mathrm{C}) = Z(40, 0) = 280 + 0 = 280$$

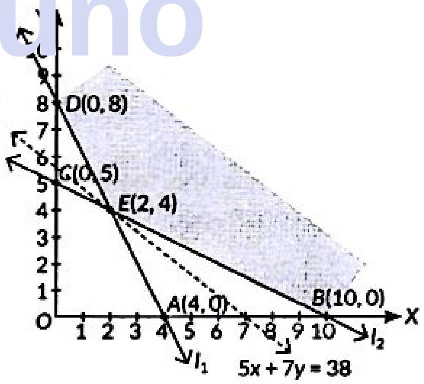
$$Z(\mathrm{D}) = Z(10, 0) = 70 + 0 = 70$$

$$\text{or Max}(=410) \text{ at } x = 30, y = 20$$

37. Soln. We have, minimize z = 5x + 7y,
Subject to constraints,
$2x + y \ge 8,\ \ x + 2y \ge 10,\ x, y \ge 0$
To solve LPP graphically, we convert inequations into equations.
Now,

$l_1 : 2x + y = 8,\ l_2 : x + 2y = 10\ and\ x = 0,\ y = 0$

l_1 and l_2 intersect at E(2, 4)
Let us draw the graph of these equations as shown below.



The corner points of the feasible region are D(0, 8), B(10, 0) and E(2, 4).

Corner points	Value of z = 5x + 7y
D (0, 8)	56
B (10, 0)	50
E (2, 4)	38 (Minimum)

From the table, we find that 38 is the minimum value of z at E(2, 4). Since the region is unbounded, so we draw the graph of inequality 5x + 7y < 38 to check whether the resulting open half plane has any point common with the feasible region. Since it has no point in common. So, the minimum value of z is obtained at E(2, 4) and the minimum value of z = 38.

38. Soln. We have, Minimise Z = 5x + 10y,
Subject to constraints:

$x+2y \le 120$

$x+y \ge 60$

$x-2y \ge 0$

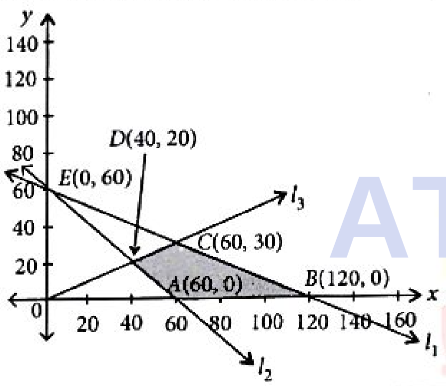
$and \quad x, y \ge 0$

To solve L.P.P graphically, we convert inequations into equations.

$l_1 : x+2y=120,\ l_2 : x+y=60,\ l_3 : x-2y=0$ $and\ \ x=0,\ y=0$

l_1 and l_2 intersect at E(0, 60), l_1 and l_3 intersect at C(60, 30), l_2 and l_3 intersect at D(40, 20).

The shaded region ABCD is the feasible region and is bounded. The corner points of the feasible region are A(60, 0), B(120, 0), C(60, 30) and D(40, 20).



Corner points	Value of Z = 5x + 10y
A(60, 0)	300 ← (Minimum)
A(120,0)	600
C(60, 30)	600
D(40, 20)	400

Hence, Z is minimum at A(60, 0) i.e., 300.

39. Soln. Maximise Z = x + 2y,
Subject to constraints:

$x+2y \ge 100,\ 2x-y<0,$

$2x+y \le 200\ and\ \ x, y \ge 0.$

Converting the inequations into equations, we obtain the lines

$l_1 : x+2y=100$ ....(i)

$l_2 : 2x-y=0$ ....(ii)

$l_3 : 2x+y=200$ ....(iii)

$l_4 : x=0$ ....(iv)

and $l_5 : y=0$ ....(v)

By intercept form, we get

$l_1 : \dfrac{x}{100}+\dfrac{y}{50}=1$

⇒ The line l_1 meets the coordinate axes at (100, 0) and (0, 50).

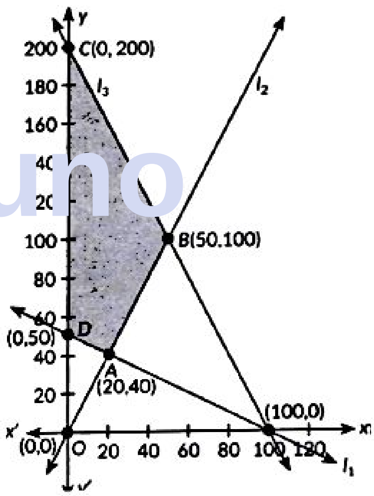
$l_2 : 2x = y$

⇒ The line l_2 passes through origin and (50, 100).

$l_3 : \dfrac{x}{100}+\dfrac{y}{200}=1$

⇒ The line l_3 meets the coordinates axes at (100, 0) and (0, 200).

$l_4 : x=0$ is the y-axis, $l_5 : y=0$ is the x-axis.



Now, plotting the above points on the graph, we get the feasible region of the LPP as shaded region ABCD. The coordinates of the corner points of the feasible region ABCD are A(20, 40), B(50, 100), C(0, 200), D(0, 50),

Now, $Z_A = 20+2\times 40 = 100$

$Z_B = 50+2\times 100 = 250,\ Z_C = 0+2\times 200 = 400$

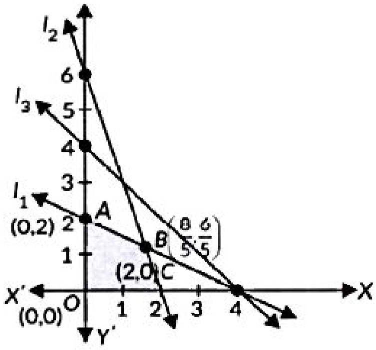
$Z_D = 0+2\times 50 = 100$

∴ Z is maximum at C(0, 200) and having value 400.

40. Soln. Let

$l_1 : 2x+4y=8,\ l_2 : 3x+y=6,\ l_3 : x+y=4;\ x=0,\ y=0$

Solving l_1 and l_2 we get $B\left(\frac{8}{5},\frac{6}{5}\right)$



Shaded portion OABC is the feasible region, where coordinates of the corner points are O(0, 0), A(0, 2).

$B\left(\frac{8}{5},\frac{6}{5}\right)$, $C(2,0)$

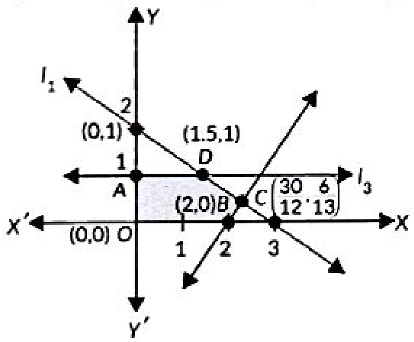
The value of objective function at these points are:

Corner Points	Value of the objective function z = 2x + 5y
O(0, 0)	2 x 0 + 5 x 0 = 0
A(0, 2)	2 x 0 + 5 x 2 = 10 (Maximum)
$B\left(\frac{8}{5},\frac{6}{5}\right)$	$2\times\frac{8}{5}+5\times\frac{6}{5}=9.2$
C(2, 0)	$2\times2+5\times0=4$

∴ The maximum value of z is 10, which is at A(0, 2).

41. Soln. Let

$l_1: 2x+3y=6$, $l_2: 3x-2y=6$, l_3 : y = 1 : x = 0, y = 0



Solving l_1 and l_3, we get D (1.5, 1)

Solving l_1 and l_2, we get $C\left(\frac{30}{13},\frac{6}{13}\right)$

Shaded portion OADCB is the feasible region, where coordinates of the corner points are O(0, 0), A(0, 1), D(1.5, 1), $C\left(\frac{30}{13},\frac{6}{13}\right)$, $B(2,0)$.

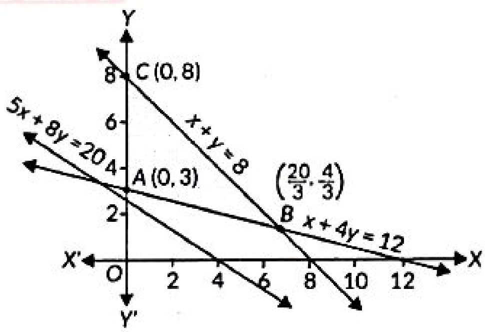
The value of the objective function at these points are:

Corner Points	Value of the objective function $z=8x+9y$
O(0, 0)	$8\times0+9\times0=0$
A(0, 1)	$8\times0+9\times1=9$
D(1.5, 1)	$8\times1.5+9\times1=21$
$C\left(\frac{30}{13},\frac{6}{13}\right)$	$8\times\frac{30}{13}+9\times\frac{6}{13}=22.6$ (Maximum)
B(2, 0)	$8\times2+9\times0=16$

The maximum value of z is 22.6, which is at $C\left(\frac{30}{13},\frac{6}{13}\right)$.

42. Soln. Converting the given inequation into equation, we get x + y = 8, x + 4y = 12, 5x + 8y = 20

Let us draw the graph of these equations as shown below



The point of intersection of the lines x + 4y = 12 and $x+y=8$ is $B=\left(\frac{20}{3},\frac{4}{3}\right)$

We have, corner points A(0, 3), $B\left(\frac{20}{3},\frac{4}{3}\right)$ and C(0, 8).

Now, Z = 30x + 20y

∴ $Z(0,3)=30(0)+20(3)=60$

$Z\left(\frac{20}{3},\frac{4}{3}\right)=30\left(\frac{20}{3}\right)+20\left(\frac{4}{3}\right)=226.6$

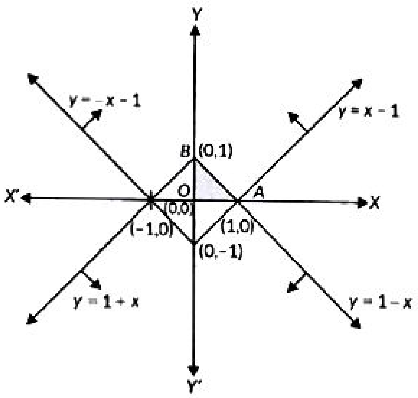
$Z(0,8) = 30(0) + 20(8) = 160$

$\therefore$ Minimum value of Z is 60 which is attained at point7 A(0, 3).

43. Soln. Max z = x + y

Subject to

$$y \geq \begin{cases} x-1, & x \geq 0 \\ -x-1, & x < 0 \end{cases}, \quad y \leq \begin{cases} 1-x, & x \geq 0 \\ 1+x, & x < 0 \end{cases} \quad and \; x, y \geq 0$$



The common region OAB is showing with shades.

Corner Points	Value of z = x + y
O(0, 0)	$z = 0 + 0 = 0$
A(1, 0)	$z = 1 + 0 = 1$ (Maximum)
B(0,1)	$z = 0 + 1 = 1$ (Maximum)

From table, maximum value of z = 1.

SURE SHOT QUESTIONS

ARVIND ACADEMY

Chapter – 13 (Solution)

Probability

➢ MCQ (1 mark)

1. Soln. (c): Given, $P(A \cap B) = \frac{7}{10}$ and $P(A) = \frac{4}{5}$.

Now, $P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{7/10}{4/5} = \frac{7}{10} \times \frac{5}{4} = \frac{7}{8}$

2. Soln. (a): Given, $P(A \cap B) = \frac{7}{10}$ and $P(B) = \frac{17}{20}$.

Now, $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{7/10}{17/20} = \frac{7}{10} \times \frac{20}{17} = \frac{14}{17}$

3. Soln. (d): Given, $P(A) = \frac{3}{10}, P(B) = \frac{2}{5}$ and

$P(A \cup B) = \frac{3}{5}$

Clearly, $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$= \frac{3}{10} + \frac{2}{5} - \frac{3}{5} = \frac{3+4-6}{10} = \frac{1}{10}$

Now, $P(B \mid A) + P(A \mid B) = \frac{P(A \cap B)}{P(A)} + \frac{P(A \cap B)}{P(B)}$

$= \frac{1/10}{3/10} + \frac{1/10}{2/5} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$

4. Soln. (c): Given, $P(A) = \frac{2}{5}, P(B) = \frac{3}{10}$ and

$P(A \cap B) = \frac{1}{5}$.

Clearly, $P(A') = 1 - P(A) = 1 - \frac{2}{5} = \frac{3}{5}$

$P(B') = 1 - P(B) = 1 - \frac{3}{10} = \frac{7}{10}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{2}{5} + \frac{3}{10} - \frac{1}{5}$

$= \frac{4+3-2}{10} = \frac{1}{2}$

And

$P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B) = 1 - \frac{1}{2} = \frac{1}{2}$

Now, $P(A' \mid B').P(B' \mid A') = \frac{P(A' \cap B')}{P(B')} . \frac{P(A' \cap B')}{P(A')}$

$= \frac{1/2}{7/10} . \frac{1/2}{3/5} = \frac{25}{42}$

5. Soln. (c): Given, $P(A) = \frac{1}{2}, P(B) = \frac{1}{3}$ and

$P(A \mid B) = \frac{1}{4}$.

Clearly, $P(A \cap B) = P(A \mid B) P(B) = \frac{1}{4} . \frac{1}{3} = \frac{1}{12}$

And $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{2} + \frac{1}{3} - \frac{1}{12} = \frac{6+4-1}{12} = \frac{9}{12} = \frac{3}{4}$

Now,

$P(A' \cap B') = P((A \cup B)')$

$= 1 - P(A \cup B) = 1 - \frac{3}{4} = \frac{1}{4}$

6. Soln. (d): Given, P(A = 0.4, P(B) = 0.8 and P(B|A) = 0.6.

Clearly, $P(A \cap B) = P(B \mid A) P(A) = 0.6 \times 0.4 = 0.24$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.4 + 0.8 - 0.24 = 0.96$

7. If A and B are two events and $A \neq \phi, B \neq \phi$, then

Soln. (b): By multiplication theorem,

$$P(A\cap B) = P(A\,|\,B)\times P(B) = P(B\,|\,A)\times P(A)$$

$$\Rightarrow \quad P(A\,|\,B) = \frac{P(A\cap B)}{P(B)}$$

8. Soln. (d): Given, P(A) = 0.4, P(B) = 0.3 and $P(A\cup B) = 0.5$.

Clearly, $P(A\cap B) = P(A) + P(B) - P(A\cup B)$

$= 0.4 + 0.3 - 0.5 = 0.2$

Now,

$$P(B'\cap A) = P(A) - P(A\cap B) = 0.4 - 0.2 = 0.2 = \frac{1}{5}$$

9. Soln. (c): Given, $P(B) = \frac{3}{5}$, $P(A\,|\,B) = \frac{1}{2}$ and

$P(A\cup B) = \frac{4}{5}$

Clearly, $P(A\cap B) = P(A\,|\,B)\,P(B) = \frac{1}{2}.\frac{3}{5} = \frac{3}{10}$

Now, as $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$$\therefore\ \frac{4}{5} = P(A) + \frac{3}{5} - \frac{3}{10} \Rightarrow\ P(A) = \frac{4}{5} - \frac{3}{10} = \frac{1}{2}$$

10. Soln. (d): From Answer 9, we have $P(A) = \frac{1}{2}$

$$\therefore \qquad P(A') = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$$

Also, we have $P(B\cap A) = \frac{3}{10}$

Now, as $P(B\cap A') + P(B\cap A) = P(B)$

[$\because$ $A'\cap B$ *and* $A\cap B$ are mutually exclusive events]

$$\therefore\ P(B\cap A') = P(B) - P(B\cap A) = \frac{3}{5} - \frac{3}{10} = \frac{3}{10}$$

$$\therefore\ P(B\,|\,A') = \frac{P(B\cap A')}{P(A')} = \frac{3/10}{1/2} = \frac{3}{5}$$

11. Soln. (d): Given, $P(B) = \frac{3}{5}$, $P(A\,|\,B) = \frac{1}{2}$ and

$P(A\cup B) = \frac{4}{5}$

Clearly, $P(A\cap B) = P(A\,|\,B)\,P(B) = \frac{1}{2}.\frac{3}{5} = \frac{3}{10}$

And $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$$\Rightarrow\ \frac{4}{5} = P(A) + \frac{3}{5} - \frac{3}{10} \Rightarrow P(A) = \frac{4}{5} - \frac{3}{10} = \frac{1}{2}$$

$$\Rightarrow\ P(A') = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$$

Also, we know, $P(A\cap B) + P(A'\cap B) = P(B)$

[As $A\cap B$ *and* $A'\cap B$ are mutually exclusive events]

$$\therefore\ \frac{3}{10} + P(A'\cap B) = \frac{3}{5} \Rightarrow\ P(A'\cap B) = \frac{3}{5} - \frac{3}{10} = \frac{3}{10}$$

Now, $P(A'\cup B) = P(A') + P(B) - P(A'\cap B)$

$$= \frac{1}{2} + \frac{3}{5} - \frac{3}{10} = \frac{5+6-3}{10} = \frac{4}{5}$$

And $P((A\cup B)') = 1 - P(A\cup B) = 1 - \frac{4}{5} = \frac{1}{5}$

12. Soln. (d): Given, $P(A) = \frac{7}{13}, P(B) = \frac{9}{13}$ and

$P(A\cap B) = \frac{4}{13}$.

We know, $P(A\cap B) + P(A'\cap B) = P(B)$

[As $A\cap B$ *and* $A'\cap B$ are mutually exclusive events.]

$$\therefore\ P(A'\cap B) = P(B) - P(A\cap B) = \frac{9}{13} - \frac{4}{13} = \frac{5}{13}$$

Now, $P(A'\,|\,B) = \frac{P(A'\cap B)}{P(B)} = \frac{5/13}{9/13} = \frac{5}{9}$

13. Soln. (c): By definition, $P(A'\,|\,B') = \frac{P(A'\cap B')}{P(B')}$

$$= \frac{P((A\cup B)')}{P(B')} = \frac{1 - P(A\cup B)}{P(B')}$$

14. Soln. (d): Given, $P(A) = \frac{3}{5}$ and $P(B) = \frac{4}{9}$.

Since, A and B are independent events, therefore

$$P(A\cap B)=P(A)P(B)=\frac{3}{5}.\frac{4}{9}=\frac{4}{15}$$

Now, $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$$=\frac{3}{5}+\frac{4}{9}-\frac{4}{15}=\frac{27+20-12}{45}=\frac{7}{9}$$

$$\therefore\ P(A'\cap B')=P((A\cup B)')=1-P(A\cup B)$$

$$=1-\frac{7}{9}=\frac{2}{9}$$

15. Soln. (d)

16. Soln. (d): Given,

$$P(A)=\frac{3}{8},\ P(B)=\frac{5}{9}\ and\ P(A\cup B)=\frac{3}{4}.$$

Clearly, $P(A\cap B)=P(A)+P(B)-P(A\cup B)$

$$=\frac{3}{8}+\frac{5}{8}-\frac{3}{4}=\frac{3+5-6}{8}=\frac{1}{4}$$

Also, we know that $P(A'\cap B)+P(A\cap B)=P(B)$

[As $A'\cap B\ and\ A\cap B$ are mutually exclusive events]

$$\therefore\ P(A'\cap B)=P(B)-P(A\cap B)=\frac{5}{8}-\frac{1}{4}=\frac{3}{8}$$

Now, $P(A\mid B).P(A'\mid B)=\dfrac{P(A\cap B)}{P(B)}.\dfrac{P(A'\cap B)}{P(B)}$

$$=\frac{1/4}{5/8}.\frac{3/8}{5/8}=\frac{3}{32}\times\frac{64}{25}=\frac{6}{25}$$

17. Soln. (c): If A and B are independent events, then

$$P(A\cap B)=P(A).P(B)$$

18. Soln. (c): Since, E and F are independent events.

$$\therefore\quad P(E\cap F)=P(E)\ P(F)$$

$$\Rightarrow\quad P(E\mid F)=P(E)\text{and }P(F\mid E)=P(F)$$

Now, $P(E\cup F)=P(E)+P(F)-P(E\cap F)$

$$\Rightarrow\quad 0.5=0.3+P(F)-0.3P(F)$$

$$\Rightarrow\quad P(F)(1-0.3)=0.5-0.3$$

$$\Rightarrow\quad P(F)=\frac{0.2}{0.7}=\frac{2}{7}$$

$$\therefore\quad P(E\mid F)-P(F\mid E)=P(E)-P(F)=0.3-\frac{2}{7}$$

$$=\quad \frac{3}{10}-\frac{2}{7}=\frac{1}{70}$$

19. Soln. (c): Required probability = $P\{(RBB),(BRB),(BBR)\}$

$$=P(RBB)+P(BRB)+P(BBR)$$

$$=\frac{5}{8}\times\frac{3}{7}\times\frac{2}{6}+\frac{3}{8}\times\frac{5}{7}\times\frac{2}{6}+\frac{3}{8}\times\frac{2}{7}\times\frac{5}{6}=3\times\frac{5}{56}=\frac{15}{56}$$

20. Soln. (b): Since, the first ball drawn is red, so we are left with 4 red and 3 blue balls. Now, we have to draw two balls out of which one should be red and other should be blue.

$\therefore$ Required probability = $P\{(RB),(BR)\}=P(RB)+P(BR)$

$$=\frac{4}{7}.\frac{3}{6}+\frac{3}{7}.\frac{4}{6}=2\times\frac{2}{7}=\frac{4}{7}$$

21. Soln. (b): Required probability

$$=P(A'\cap B\cap C)+P(A\cap B'\cap C)+P(A\cap B\cap C')$$

$$=(1-P(A))P(B)P(C)+P(A)(1-P(B))P(C)+P(A)P(B)(1-P(C))$$

$$=(1-0.4)(0.3)(0.2)+(0.4)(1-0.3)(0.2)+(0.4)(0.3)(1-0.2)$$

$$=0.036+0.056+0.096=0.188$$

22. Soln. (d): Sample space, S = {BBB, BBG, BGB, GBB, GGB, GBG, BGG, GGG}

Let E_1 be the event that eldest child is a girl and E_2 be the event that atleast one child is a girl. Then,

$$E_1=\{GBB,GGB,GBG,GGG\}\Rightarrow P(E_1)=\frac{4}{8}$$

$$E_2=\{BBG,BGB,GBB,GGB,GBG,BGG,GGG\}$$

$$\Rightarrow\ P(E_2)=\frac{7}{8}$$

$$and\ \ E_1\cap E_2=\{GBB,GGB,GBG,GGG\}$$

$$P(E_1\cap E_2)=\frac{4}{8}$$

Now, required probability = $P(E_1\mid E_2)=\dfrac{P(E_1\cap E_2)}{P(E_2)}$

$$=\frac{4/8}{7/8}=\frac{4}{7}$$

23. Soln. (c): Sample space for die are {1, 2, 3, 4, 5, 6}

And even numbers on die are {2, 4, 6}

$\therefore$ Probability of getting an even number = $\frac{3}{6}=\frac{1}{2}$

Total number of cads = 52

And number of spade cards = 13

$\therefore$ Probability of getting a spade card = $\frac{13}{52}=\frac{1}{4}$

Hence, required probability

= P (an even number on die) x P(a spade card) = $\frac{1}{2}\times\frac{1}{4}=\frac{1}{8}$

24. Soln. (a): Required probability = P{GGB, GBG, BGG} = P(GGB) + P(GBG) + P(BGG)

$$=\frac{3}{8}\times\frac{2}{7}\times\frac{2}{6}+\frac{3}{8}\times\frac{2}{7}\times\frac{2}{6}+\frac{2}{8}\times\frac{3}{7}\times\frac{2}{6}=3\times\frac{1}{28}=\frac{3}{28}$$

25. Soln. (d): Required probability = P(DD) = $\frac{3}{8}\times\frac{2}{7}=\frac{3}{28}$

26. Soln. (b): Throwing of eight coins simultaneously is the same as throwing of one coin 8 times.

Now, probability of getting a head on tossing a coin, $p=\frac{1}{2}$

$$\Rightarrow\ q=1-p=1-\frac{1}{2}=\frac{1}{2}$$

Thus, we have a binomial distribution with $p=\frac{1}{2}, q=\frac{1}{2}$ and n = 8.

Hence, required probability = ${}^8C_3p^3q^5$

$$=56\times\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^5=56\times\frac{1}{8}\times\frac{1}{32}=\frac{7}{32}$$

27. Soln. (c): Let E_1 be the event that sum is less than 6 and E_2 be the event that sum is 3.

Then, E_1 = {(1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (1, 4), (4, 1), (2, 3), (3, 2), (2, 2)}

And E_2 = {(1, 2), (2, 1)}

$$\therefore\ E_1\cap E_2=\{(1,2),(2,1)\}$$

Now, $P(E_1)=\frac{10}{36}=\frac{5}{18}$ and $P(E_1\cap E_2)=\frac{2}{36}=\frac{1}{18}$

Hence, required probability = $P(E_2|E_1)$

$$=\frac{P(E_1\cap E_2)}{P(E_1)}=\frac{1/18}{5/18}=\frac{1}{5}$$

28. Soln. (c): For a Binomial distribution, outcomes at different trials must be independent.

29. Soln. (a): Total number of cards = 52

Number of queens = 4

Hence, required probability = $\frac{4}{52}\times\frac{4}{51}=\frac{1}{13}\times\frac{1}{13}$

30. Soln. (b): It is a binomial distribution case with n = 10 and p = probability of guessing correctly = $\frac{1}{2}$

$$\Rightarrow\ q=1-p=1-\frac{1}{2}=\frac{1}{2}$$

Now, required probability = $P(X\geq 8)$

$$=P(X=8)+P(X=9)+P(X=10)$$

$$={}^{10}C_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2+{}^{10}C_9\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)+{}^{10}C_{10}\left(\frac{1}{2}\right)^{10}$$

$$=\left(\frac{1}{2}\right)^{10}\left[{}^{10}C_8+{}^{10}C_9+{}^{10}C_{10}\right]=\left(\frac{1}{2}\right)^{10}\times[45+10+1]$$

$$=56\times\left(\frac{1}{2}\right)^{10}=\frac{7}{128}$$

31. Soln. (a): It follows a Binomial distribution with n = 5, p = probability that chosen person is a swimmer = 1 − 0.3 = 0.7 and q = 0.3

Hence, required probability = P(X = 4)

$$={}^5C_4p^4q^1={}^5C_4(0.7)^4(0.3)$$

32. Soln. (c): We know $\sum p_i=1$

$$\therefore \frac{5}{k}+\frac{7}{k}+\frac{9}{k}+\frac{11}{k}=1 \Rightarrow \frac{32}{k}=1 \Rightarrow k=32$$

33. Soln. (d):

$$E(X)=\sum x_i p_i=(-4)\times 0.1+(-3)\times 0.2+(-2)\times 0.3+(-1)\times 0.2+0\times 0.2$$

$$= 0.4-0.6-0.6-0.2=-1.8$$

34. Soln. (d): $E(X^2)=\sum x_i^2 pi$

$$=(1)^2\times\frac{1}{10}+(2)^2\times\frac{1}{5}+(3)^2\times\frac{3}{10}+(4)^2\times\frac{2}{5}$$

$$=\frac{1}{10}+\frac{4}{5}+\frac{27}{10}+\frac{32}{5}=\frac{1+8+27+64}{10}=\frac{100}{10}=10$$

35. Soln. (a): We have

$\dfrac{P(x=r)}{P(x=n-r)}$ is independent of n and r

i.e. $\dfrac{{}^nC_r p^r q^{n-r}}{{}^nC_{n-r} p^{n-r} q^r}$ is independent of n and r

$\Rightarrow \dfrac{{}^nC_r q^{n-2r}}{{}^nC_r p^{n-2r}}$ is independent of n and r.

$$[\because {}^nC_r={}^nC_{n-r}]$$

$\Rightarrow \dfrac{q^{n-2r}}{p^{n-2r}}$ is independent of n and r

$$\Rightarrow \frac{q}{p}=1 \Rightarrow \frac{1-p}{p}=1 \Rightarrow 1-p=p$$

$$\Rightarrow 2p=1 \Rightarrow p=\frac{1}{2}$$

36. Soln.

(c): Since each coin turns up on either a head or tail.

$\therefore$ Total possible outcomes = 2^5 = 32

Let A be the event that all tails comes up.

$\therefore$ n(A) = 1 {i.e., (T, T, T, T, T)

So, required probability = $1-P(A)=1-\dfrac{1}{32}=\dfrac{31}{32}$

37. Soln. (d): Here, A = {4, 5, 6}, B = {1, 2, 3, 4}

$A\cap B=\{4\}$

Now, $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$$=\frac{3}{6}+\frac{4}{6}-\frac{1}{6}=1$$

38. Soln. (d): We have, P(A) = 0.4, P(B) = 0.8 and P(B/A) = 0.6

We know that $P(B/A)=\dfrac{P(A\cap B)}{P(A)}$

$$\Rightarrow 0.6=\frac{P(A\cap B)}{0.4}$$

$$\Rightarrow P(A\cap B)=0.24$$

Now, $P(A\cup B)=P(A)+P(B)-P(A\cap B)$
$=0.4+0.8-0.24=0.96$

Hence, $P(A\cup B)=0.96$

39. Soln. (c): We know that,

$$P(B/A)=\frac{P(B\cap A)}{P(A)}=\frac{7/10}{4/5}=\frac{7}{8}$$

40. Soln. (a): Sample space = {HH, HT, TH, TT}

Let A be the event of coming up two heads

$\therefore A=\{HH\}\Rightarrow P(A)=\dfrac{1}{4}$

And B be the event of coming up atleast one head

$\therefore B=\{HH, HT, TH\}\Rightarrow P(B)=\dfrac{3}{4}$

Also, $A\cap B=\{HH\}\Rightarrow P(A\cap B)=\dfrac{1}{4}$

So, required probability =

$$P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{1}{4}}{\frac{3}{4}}=\frac{1}{3}$$

So, assertion is true,

Also, reason is true and it is the correct explanation of assertion.

41. Soln. (c): Let A be the event that the card is a spade and B be the event that the picked cad is a queen.

We have a total of 13 spades and 4 queen cards. Also only one queen is from spade.

$$\therefore\ P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{1}{52}}{\frac{4}{52}}=\frac{1}{4}$$

42. Soln. (c): Given, A and B are independent events.

Also, $P(A)=\frac{1}{3}$ *and* $P(B)=\frac{1}{4}$

Now,

$$P(B'|A)=\frac{P(B'\cap A)}{P(A)}$$

$$=\frac{P(B')P(A)}{P(A)}\quad [\because A, B \text{ are independent events}]$$

$$=P(B')=1-P(B)=1-\frac{1}{4}=\frac{3}{4}$$

➤ Assertion-Reasoning (1 mark)

43. Sol. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation: P(A B') = P(A) + P(B') - P(A)P(B')

$\therefore$ 0.8 = 0.3 + P(B') - 0.3 P(B')

$\Rightarrow$ 0.5 = P(B') (0.7)

$\Rightarrow$ P(B') = 57

$\Rightarrow$ P(B) = 1 - 57

$= \frac{2}{7}$

44. Sol. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation: Both A and R are true and R is the correct explanation of A.

45. Sol. **(d)** A is false but R is true.

Explanation: Reason: P(A ∩ $\bar{B}$) = P(A) - (A ∩ B)

Additional theorem,

⇒If A B are 2 events associated with random experiment, then P(A ∪ B) = P(A) + P(B) - P(A∩ B)

If A ⊂ B are events associated with random experiments,

⇒P(A∪ B∪ C) = P(A) + P(B) + P(C) - P(A ∩ B) - P(B ∩C) - P(A∩C) + P(A ∩ B ∩ C)

If A, B, C are mutually exclusive,

⇒P(A∪B∪C) = P(A) + P(B) + P(C)

If any 2 events occur of A only

$\Rightarrow$ P(A ∩ $\bar{B}$) = P(A) - (A ∩ B)

$$\Rightarrow P\left(\frac{A\cap\bar{B}}{C}\right)=P\left(\frac{A}{C}\right)-P\left(\frac{A\cap B}{C}\right)$$

Hence, the answer is Assertion is incorrect but reason is correct.

46. Sol. **(c)** A is true but R is false.

Explanation: A is true but R is false.

➤ Case Study Questions

47. Soln. Let A be the event of commiting an error and E_1, E_2 and E_3 be the events that Vinay, Sonia and Iqbal processed the form.

(i) (b): Required probability = P(A|E₂)

$$=\frac{P(A\cap E_2)}{P(E_2)}=\frac{\left(0.04\times\frac{20}{100}\right)}{\left(\frac{20}{100}\right)}=0.04$$

(ii) (c): Required probability = $P(A\cap E_2)$

$$=0.04\times\frac{20}{100}=0.008$$

(iii) (b): Total probability is given by

$$P(A)=P(E_1).P(A|E_1)+P(E_2).P(A|E_2)+P(E_3).P(A|E_3)$$

$$=\frac{50}{100}\times0.06+\frac{20}{100}\times0.04+\frac{30}{100}\times0.03=0.047$$

(iv) (d): Using Bayes' theorem, we have

$$P(E_1|A)=\frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1)+P(E_2).P(A|E_2)+P(E_3).P(A|E_3)}$$

$$=\frac{0.5\times0.06}{0.5\times0.06+0.2\times0.04+0.3\times0.03}=\frac{30}{47}$$

$\therefore$ Required probability = $P(\overline{E_1}|A)$

$$=1-P(E_1|A)=1-\frac{30}{47}=\frac{17}{47}$$

(v) (d):

$$\sum_{i=1}^{3}P(E_i|A)=P(E_1|A)+P(E_2|A)+P(E_3|A)=1$$

[∵ Sum of posterior probabilities is 1]

Questions

48. Soln.

$\boxed{\text{Question : 4}}$
(option-2)

$P(\text{A hits}) = \frac{1}{3}$ (P(A))

$P(\text{B hits}) = \frac{2}{5}$ (P(B))

$P(\text{A doesn't hit}) = 1 - \frac{1}{3}$ $[P(A) + P(A') = 1]$

$P(A') = \frac{2}{3}$

$P(\text{B doesn't hit}) = 1 - \frac{2}{5}$

$P(B') = \frac{3}{5}$

As these are independent events = P(not hitting)
$= P(A').P(B')$
$[P(A \cap B) = P(A).P(B)]$

~~Let...~~

$P(\text{target is hit}) = 1 - P(\text{target not hit})$

$= 1 - P(A') \times P(B')$

$= 1 - \frac{2}{3} \times \frac{3}{5}$

$= 1 - \frac{2}{5}$

$= \frac{3}{5}$

Answer Probability of target being hit is $\frac{3}{5}$

[Topper's Answer, 2022]

49. Soln. Let E_1 be the event that bag I is chosen, E_2 be the event that bag II is chosen and A be the event that red ball is drawn.
Clearly, E_1 and E_2 are mutually exclusive and exhaustive events.
Since, one of the bag is chosen at random

$$\therefore\ P(E_1) = \frac{1}{2} \text{ and } P(E_2) = \frac{1}{2}$$

$$P(A|E_1) = \frac{1}{4} \text{ and } P(A|E_2) = \frac{3}{8}$$

By using law of total probability, we get

$$P(A) = P(E_1)P(A|E_1) + P(E_2)P(A|E_2)$$

$$= \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{3}{8} = \frac{1}{8} + \frac{3}{16} = \frac{5}{16}$$

50. Soln. Let E_1, E_2 and A denote the events defined as follow:

E_1 = selecting a purse 1

E_2 = selecting a purse 2

A = drawing a silver coin

Since one of two purses is selected randomly

$$\therefore\ P(E_1) = \frac{1}{2} \text{ and } P(E_2) = \frac{1}{2}$$

Now, $P(A/E_1) = \frac{3}{9} = \frac{1}{3}$ and $P(A/E_2) = \frac{4}{7}$

Using the total law of probability, we have
Required probability,

$$P(A) = P(E_1).P(A/E_1) + P(E_2).(A/E_2)$$

$$\Rightarrow P(A) = \frac{1}{2}.\frac{1}{3} + \frac{1}{2} \times \frac{4}{7} = \frac{1}{6} + \frac{2}{7} = \frac{19}{42}$$

51. Soln. Since, A be the event of number obtained is even
Then, A = {2, 4, 6}
And B be the event of number obtained is red then,
B = {1, 2, 3}

$\therefore$ $A \cap B = \{2\}$

So,

$$P(A) = \frac{3}{6} = \frac{1}{2};\ P(B) = \frac{3}{6} = \frac{1}{2};\ P(A \cap B) = \frac{1}{6}$$

Now, $P(A \cap B) \neq P(A).P(B)$

$$\frac{1}{6} \neq \frac{1}{4}$$

Hence, the events A and B are not independent events.

52. Soln. Let E_1, E_2 and E_3 be the events denoting the selection of A, B and C as managers respectively.

$P(E_1)$ = Probability of selection of $A = \frac{1}{7}$

$P(E_2)$ = Probability of selection of $B = \frac{2}{7}$

$P(E_3)$ = Probability of selection of $C = \frac{4}{7}$

Let A be the event denoting the change not taking place.

$P(A/E_1)$ = Probability that A does not introduce change = 0.2

$P(A/E_2)$ = Probability that B does not introduce change = 0.5

$P(A/E_3)$ = Probability that C does not introduce change = 0.7

$\therefore$ Required probability = $P(E_3/A)$

By Bayes' theorem, we have

$$P(E_3/A) = \frac{P(E_3)P(A/E_3)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3)}$$

$$= \frac{\frac{4}{7}\times 0.7}{\frac{1}{7}\times 0.2+\frac{2}{7}\times 0.5+\frac{4}{7}\times 0.7}$$

$$= \frac{2.8}{0.2+1+2.8}$$

$$= \frac{2.8}{4} = 0.7$$ Ans.

53. Soln. Total of 7 on the dice can be obtained in the following ways:

$$(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)$$

Probability of getting a total of $7 = \frac{6}{36} = \frac{1}{6}$

Probability of not getting a total of $7 = 1-\frac{1}{6} = \frac{5}{6}$

Total of 10 on the dice can be obtained in the following ways:

(4, 6), (6, 4), (5, 5)

Probability of getting a total of 10

$$= \frac{3}{36} = \frac{1}{12}$$

Probability of not getting a total of 10

$$= 1-\frac{1}{12} = \frac{11}{12}$$

Let E and F be the two events, defined as follows:

E = Getting a total of 7 in a single throw of a dice

F = Getting a total of 10 in a single throw of a dice

$$P(\text{E}) = \frac{1}{6}, P(\bar{E}) = \frac{5}{6}$$

$$P(\text{F}) = \frac{1}{12}, P(\bar{F}) = \frac{11}{12}$$

A wins if he gets a total of 7 in 1st, 3rd or 5th .... throws.

Probability of A getting a total of 7 in the 1st throw

$$= \frac{1}{6}$$

A will get the 3rd throw if he fails in the 1st throw and B fails in the 2nd throw.

Probability of A getting a total of 7 in the 3rd throw

$$= P(\bar{E})\text{P}(\bar{F})\text{P(E)} = \frac{5}{6}\times\frac{11}{12}\times\frac{1}{6}$$

Similarly, probability of getting a total of 7 in the 5th

grow = $P(\bar{E})\text{P}(\bar{F})\text{P}(\bar{E})\text{P}(\bar{E})\text{P(E)}$

$$= \frac{5}{6}\times\frac{11}{12}\times\frac{5}{6}\times\frac{11}{12}\times\frac{1}{6} \text{ and so on}$$

Probability of winning of A

$$= \frac{1}{6}+\left(\frac{5}{6}\times\frac{11}{12}\times\frac{1}{6}\right)+\left(\frac{5}{6}\times\frac{11}{12}\times\frac{5}{6}\times\frac{11}{12}\times\frac{1}{6}\right)+\ldots\ldots$$

$$= \frac{\frac{1}{6}}{1-\frac{5}{6}\times\frac{11}{12}} = \frac{12}{17}$$

$\therefore$ Probability of winning of B = 1 – Probability of winning of A

$$= 1-\frac{12}{17} = \frac{5}{17}$$

Ans.

54. Soln. $P(\text{E}) = \frac{7}{13}$

$$P(\text{F}) = \frac{9}{13}$$

And $P(\text{E}\cap\text{F}) = \frac{4}{13}$

(Given)

(i) $P(\bar{E}/\text{F}) = \dfrac{P(\bar{E}\cap\text{F})}{P(\text{F})}$

$$= \frac{P(\text{F})-\text{P}(\text{E}\cap\text{F})}{P(\text{F})}$$

$$= 1-\frac{P(\text{E}\cap\text{F})}{P(\text{F})} = 1-\frac{\frac{4}{13}}{\frac{9}{13}}$$

Or $P(\bar{E}/\text{F}) = 1-\frac{4}{9} = \frac{5}{9}$

(ii) $P(\bar{E}/\bar{F}) = \dfrac{P(\bar{E}\cap\bar{F})}{P(\bar{F})}$

$$= \frac{P(\overline{E\cup F})}{P(\bar{F})}$$

$$= \frac{1-P(\text{E}\cup\text{F})}{1-P(\text{F})}$$

$$P(\text{E}\cup\text{F}) = \text{P(E)}+\text{P(F)}-\text{P}(\text{E}\cap\text{F})$$

$$= \frac{7}{13}+\frac{9}{13}-\frac{4}{13}$$

$$= \frac{12}{13}$$

$$P(E/F) = \frac{1-\frac{12}{13}}{1-\frac{9}{13}}$$

$$= \frac{\frac{1}{13}}{\frac{4}{13}} = \frac{1}{4}$$

55. Soln. Given, E = (1, 3, 5), F = (2, 3), G = (2, 3, 4, 5)

$$P(E) = \frac{3}{6}$$

$$P(F) = \frac{2}{6}$$

$$P(G) = \frac{4}{6} = \frac{2}{3}$$

$$P(E \cup F) = \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{2}{3}$$

$$P(E \cap F) = \frac{1}{6}$$

(i) $(E \cup F) \cap G = (2, 3, 5)$

$$P[(E \cup F) \cap G] = \frac{3}{6}$$

$$P[(E \cup F/G)] = \frac{P[(E \cup F) \cap G]}{P(G)}$$

$$= \frac{\frac{3}{6}}{\frac{4}{6}} = \frac{3}{4}$$

(ii) $(E \cap F) \cap G = 3$

$$P[(E \cap F) \cap G] = \frac{1}{6}$$

$$P[(E \cap F)/G] = \frac{P[(E \cup F) \cap G]}{P(G)}$$

$$= \frac{\frac{1}{6}}{\frac{4}{6}} = \frac{1}{4}$$

56. Soln. $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

If E and F are independent, then

$$= P(E \cap F) = P(E) \times P(F)$$

$$P(E \cup F) = P(E) + P(F) - P(E) \times P(F)$$

$$P(E \cup F) = \frac{1}{2} + \frac{1}{5} - \frac{1}{2} \times \frac{1}{5}$$

$$= \frac{1}{2} + \frac{1}{5} - \frac{1}{10} = \frac{5+2-1}{10}$$

$$= \frac{6}{10} = \frac{3}{5}$$

$$P(\overline{E \cup F}) = 1 - P(E \cup F)$$

$$= 1 - \frac{3}{5} = \frac{2}{5}$$

57. Soln. Let

E be the event = A solves the problem

F be the event = B solves the problem

G be the event = C solves the problem

H be the event = D solves the problem

$$P(E) = \frac{1}{3} \qquad P(\bar{E}) = \frac{2}{3}$$

$$P(F) = \frac{1}{4} \qquad P(\bar{F}) = \frac{3}{4}$$

$$P(G) = \frac{1}{5} \qquad P(\bar{G}) = \frac{4}{5}$$

$$P(H) = \frac{2}{3} \qquad P(\bar{H}) = \frac{1}{3}$$

(i) The probability = $P(E \cup F \cup G \cup H)$

$$= 1 - P(\bar{E} \cap \bar{F} \cap \bar{G} \cap \bar{H})$$

$$= 1 - P(\bar{E}) \times P(\bar{F}) \times P(\bar{G}) \times P(\bar{H})$$

$$= 1 - \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \frac{1}{3}$$

$$= \frac{13}{15}$$

(ii) The required probability = $P(\bar{E}) \times P(\bar{F}) \times P(\bar{G})$

$$\times P(\bar{H}) + P(E) \times P(\bar{F}) \times P(\bar{G}) \times P(\bar{H}) + P(\bar{E}) \times P(F) \times P(\bar{G})$$

$$\times P(\bar{H}) + P(\bar{E}) \times P(\bar{F}) \times P(G) \times P(\bar{H}) + P(\bar{E}) \times P(\bar{F}) \times P(\bar{G}) \times P(H$$

$$= \frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}\times\frac{1}{3}+\frac{1}{3}\times\frac{3}{4}\times\frac{4}{5}\times\frac{1}{3}+\frac{2}{3}\times\frac{1}{4}\times\frac{4}{5}\times\frac{1}{3}$$

$$+\frac{2}{3}\times\frac{3}{4}\times\frac{1}{5}\times\frac{1}{3}+\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}\times\frac{2}{3}$$

$$= \frac{2}{15}+\frac{1}{15}+\frac{2}{45}+\frac{1}{30}+\frac{4}{15}$$

$$= \frac{7}{15}+\frac{1}{30}+\frac{2}{45}$$

$$= \frac{42+3+4}{90} = \frac{49}{90}$$

58\. Soln. Let A and B denote 'A speaks the truth' and 'B speaks the truth' respectively.

Now $P(\text{A}) = \frac{75}{100}$

$$P(\overline{A}) = 1 - \text{P(A)}$$

$$= 1 - \frac{75}{100} = \frac{25}{100}$$

$$P(\text{B}) = \frac{90}{100}$$

$$P(\overline{B}) = 1 - \text{P(B)}$$

$$= 1 - \frac{90}{100} = \frac{10}{100}$$

Required probability = $P(\text{A})\,\text{P}(\overline{B}) + \text{P}(\overline{A})\,\text{P(B)}$

$$= \frac{75}{100}\times\frac{10}{100}+\frac{25}{100}\times\frac{90}{100}$$

$$= \frac{30}{100} = 30\%$$

Hence, they are likely to contradict each other in 30% of the cases in stating the same fact. Though B speaks truth in 90% of the cases but he also lies in 10% of the cases. So, his statement is not always true.

59\. Soln. No. of coins with head on both sides = (n – 1)

No. of fair coins = (n + 2)

Let event

E_1 = Picking a coin with head on both sides

E_2 = Picking a fair coin

A : getting a head on tossing the coin

$$P(\text{E}_1) = \frac{n-1}{2n+1},$$

$$P(\text{E}_2) = \frac{n+2}{2n+1}$$

$$P(\text{E}_1) = 1,\ \text{P(A/ E}_2) = 1/2$$

$$\text{P(A)} = \text{P(E}_1)\text{P(A/ E}_1) + \text{P(E}_2)\text{P(A/ E}_2)$$

$$= \frac{n-1}{2n+1}.1 + \frac{n+2}{2n+1}.\frac{1}{2}$$

$$= \frac{3n}{2(2\text{n}+1)}$$

$$or \quad \frac{3n}{2(2\text{n}+1)} = \frac{31}{42}\ or\ n = 31.$$

60\. Soln. Let A_i and B_i be the events of throwing 10 by A and B in the respectively i^{th} turn, then

$$P(\text{A}_i) = \text{P(B}_i) = \frac{1}{12}$$

And $P(\overline{A}_i) = \text{P}(\overline{B}_i) = \frac{11}{12}$

Probability of wining A, when A starts first

$$= \frac{1}{12} + \left(\frac{11}{12}\right)^2\frac{1}{12} + \left(\frac{11}{12}\right)^4\frac{1}{12} + \ldots\ldots$$

$$= \frac{\frac{1}{12}}{1-\left(\frac{11}{12}\right)^2} = \frac{12}{23}$$

Probability of winning of

$$B = 1 - P(\text{A}) = 1 - \frac{12}{23} = \frac{11}{23}$$

61\. Soln. Let E_1 = Selecting bag A

And E_2 = Selecting bag B

$$\therefore \quad P(\text{E}_1) = \frac{1}{3}$$

$$and \quad P(\text{E}_2) = \frac{2}{3}$$

Let A = Getting one red and one black ball

$$\therefore\ P(\text{A/ E}_1) = \frac{{}^4C_1.{}^6C_1}{{}^{10}C_2} = \frac{8}{15}$$

And $(\text{A/ E}_2) = \frac{{}^7C_1.{}^3C_1}{{}^{10}C_2} = \frac{7}{15}$

$$P(\text{A}) = \text{P(E}_1).\text{P(A/ E}_1) + \text{P(E}_2).\text{P(A/ E}_2)$$

$$= \frac{1}{3}.\frac{8}{15} + \frac{2}{3}.\frac{7}{15}$$
$$= \frac{22}{45}$$

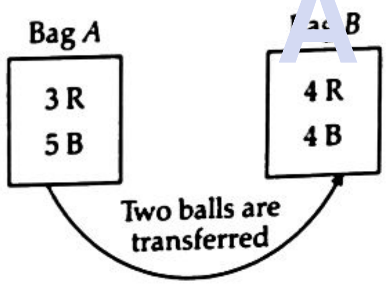
62. Soln. Let E₁ : Selecting a student with 100% attendance

E₂ : Selecting a student who is not regular

A : selected student attains A grade.

$$P(E_1) = \frac{30}{100} \text{ and } P(E_2) = \frac{70}{100}$$
$$P(A/E_1) = \frac{70}{100} \text{ and } P(A/E_2) = \frac{10}{100}$$
$$P(E_1/A) = \frac{P(E_1).P(A/E_1)}{P(E_1).P(A/E_1) + P(E_2)P(A/E_2)}$$
$$= \frac{\frac{30}{100} \times \frac{70}{100}}{\frac{30}{100} \times \frac{70}{100} + \frac{70}{100} \times \frac{10}{100}} = \frac{3}{4}$$

63. Soln.



Let us define the following events:

E_1 = one red and one black ball is transferred

E_2 = two red balls are transferred

E_3 = two black balls are transferred

E = drawn ball is red.

Then,

$$P(E_1) = \frac{{}^3C_1 \times {}^5C_1}{{}^8C_2} = \frac{3 \times 5}{28} = \frac{15}{28}$$
$$P(E_2) = \frac{{}^3C_2}{{}^8C_2} = \frac{3}{28}$$
$$P(E_3) = \frac{{}^5C_2}{{}^8C_2} = \frac{10}{28}$$
$$P\left(\frac{E}{E_1}\right) = \frac{5}{10},\ P\left(\frac{E}{E_2}\right) = \frac{6}{10},\ P\left(\frac{E}{E_3}\right) = \frac{4}{10}$$

Now, required probability = $P\left(\frac{E_2}{E}\right)$

$$= \frac{P(E_2).P\left(\frac{E}{E_2}\right)}{P(E_1).P\left(\frac{E}{E_1}\right) + P(E_2).P\left(\frac{E}{E_2}\right) + P(E_3).P\left(\frac{E}{E_3}\right)}$$
$$= \frac{\frac{3}{28}.\frac{6}{10}}{\frac{15}{28}.\frac{5}{10} + \frac{3}{28}.\frac{6}{10} + \frac{10}{28}.\frac{4}{10}}$$
$$= \frac{18}{75 + 18 + 40} = \frac{18}{133}$$

64. Soln. Let E₁ : Event that the selected bolt is manufactured by machine A,

E₂ : Event that the selected bolt is manufactured by machine B,

E₃ : Event that the selected bolt is manufactured by machine C,

And E : Event that the selected bolt is defective.

Then, we have

$$P(E_1) = 30\% = \frac{30}{100}$$
$$P(E_2) = 50\% = \frac{50}{100}$$
$$\text{and } P(E_3) = 20\% = \frac{20}{100}$$

Also, given that 3%, 4% and 1% bolts manufactured by machines A, B and C respectively are defective. So,

$$P\left(\frac{E}{E_1}\right) = 3\% = \frac{3}{100}$$
$$P\left(\frac{E}{E_2}\right) = 4\% = \frac{4}{100}$$
$$P\left(\frac{E}{E_3}\right) = 1\% = \frac{1}{100}$$

Now, the probability that selected bolt which is defective, is manufactured by machine B

$$= P\left(\frac{E_2}{E}\right) = \frac{P(E_2).P\left(\frac{E}{E_2}\right)}{P(E_1).P\left(\frac{E}{E_1}\right) + P(E_2).P\left(\frac{E}{E_2}\right)} + P(E_3).P\left(\frac{E}{E_3}\right)$$

$$= \frac{\frac{50}{100} \times \frac{4}{100}}{\frac{30}{100} \times \frac{3}{100} + \frac{50}{100} \times \frac{4}{100} + \frac{20}{100} \times \frac{1}{100}}$$

$$= \frac{200}{90+200+20} = \frac{200}{310}$$

$\therefore$ The probability that selected bolt which is defective, is not manufactured by machine B

$$= 1 - P\left(\frac{E_2}{E}\right) = 1 - \frac{200}{310} = \frac{110}{310} = \frac{11}{31}$$

65. Soln. Let A be the event that the picked up tube is defective.

Let A_1, A_2, A_3 be events such that

A_1 = event of producing tube by machine E_1

A_2 = event of producing tube by machine E_2

A_3 = event of producing tube by machine E_3

$$P(A_1) = \frac{50}{100} = \frac{1}{2}, \qquad P(A_2) = \frac{25}{100} = \frac{1}{4}, \qquad P(A_3) = \frac{25}{100} = \frac{1}{4}$$

Also, $P\left(\frac{A}{A_1}\right) = \frac{4}{100} = \frac{1}{25}$

$$P\left(\frac{A}{A_2}\right) = \frac{4}{100} = \frac{1}{25} \quad and \quad P\left(\frac{A}{A_3}\right) = \frac{5}{100} = \frac{1}{20}$$

Now, P(A) is required.

From concept of total probability,

$$P(A) = P(A_1).P\left(\frac{A}{A_1}\right) + P(A_2).P\left(\frac{A}{A_2}\right) + P(A_3).P\left(\frac{A}{A_3}\right)$$

$$= \frac{1}{2} \times \frac{1}{25} + \frac{1}{4} \times \frac{1}{25} + \frac{1}{4} \times \frac{1}{20} = \frac{1}{50} + \frac{1}{100} + \frac{1}{80}$$

$$= \frac{8+4+5}{400} = \frac{17}{400} = 0.0425$$

66. Soln. The probability distribution of X is:

X	0	1	2	3	4
P(X)	0.1	K	2k	2k	k

(a) We know that $\sum_{i=1}^{n} p_i = 1$

Therefore $0.1 + k + 2k + 2k + k = 1$

i.e., $k = 0.15$

(b) P(you study at least two hours) = $P(X \geq 2)$

$$= P(X = 2) + P(X = 3) + P(X = 4)$$

$$= 2k + 2k + k = 5k = 5 \times 0.15 = 0.75$$

P(you study exactly two hours) = P (X = 2) = 2k = 2 x 0.15 = 0.3

P(you study at most two hours) = $P(X \leq 2)$

$$= P(X = 0) + P(X = 1) + P(X = 2)$$

$$= 0.1 + k + 2k = 0.1 + 3k = 0.1 + 3 \times 0.15 = 0.55$$

67. Soln. Let, E_1 : Event that lost card is a spade

E_2 : Event that lost card is a not spade

A : Event that three spades are drawn without replacement from 51 cards

$$P(E_1) = \frac{13}{52} = \frac{1}{4}, \qquad P(E_2) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P(A/E_1) = \frac{^{12}C_3}{^{51}C_3}, \quad P(A/E_2) = \frac{^{13}C_3}{^{51}C_3},$$

$$P(E_1/A) = \frac{\frac{1}{4}.\frac{^{12}C_3}{^{51}C_3}}{\frac{1}{4}.\frac{^{12}C_3}{^{51}C_3} + \frac{3}{4}.\frac{^{13}C_3}{^{51}C_3}}$$

$$= \frac{10}{49}$$

68. Soln. Let E_1 : selected student is a hosteler

E_2 : selected student is a day scholar

A : selected student attain 'A' grade in exam.

$$P(E_1) = \frac{60}{100}, \ P(E_2) = \frac{40}{100}$$

$$P(A/E_1) = \frac{30}{100}, \ P(A/E_2) = \frac{20}{100}$$

$$P(E_1/A) = \frac{P(E_1).P(A/E_1)}{P(E_1).P(A/E_1) + P(E_2).P(A/E_2)}$$

$$= \frac{\frac{60}{100}.\frac{30}{100}}{\frac{60}{100}.\frac{30}{100} + \frac{40}{100}.\frac{20}{100}} = \frac{9}{13}$$

69. Soln. Total number of honest people = 30

The number of people who speak truth = 20

The number of people who do not speak truth = 30 – 20 = 10

Number of selected persons = 2

Let X denote the number of people who speak truth.

$\therefore$ X can take values 0, 1, 2

Now,

$$P(X=0)=\frac{{}^{20}C_0\times{}^{10}C_2}{{}^{30}C_2}=\frac{10\times 9}{30\times 29}=\frac{9}{87}$$

$$P(X=1)=\frac{{}^{20}C_1\times{}^{10}C_1}{{}^{30}C_2}=\frac{20\times 10\times 2}{30\times 29}=\frac{40}{87}$$

$$P(X=2)=\frac{{}^{20}C_2\times{}^{10}C_0}{{}^{30}C_2}=\frac{20\times 19}{30\times 29}=\frac{38}{87}$$

Hence, the probability distribution of X is

X	0	1	2
P(X)	$\frac{9}{87}$	$\frac{40}{87}$	$\frac{38}{87}$

Mean of the distribution

$$\bar{X}=0\times\frac{9}{87}+1\times\frac{40}{87}+2\times\frac{38}{87}=\frac{40+76}{87}=\frac{116}{87}=\frac{4}{3}$$

Since, out of 30 honest people, 20 always speak truth. So, the value of truthfulness and morality is described here.

70. Soln. Let X be the random variable.

$\therefore$ X can take values 2, 3, 4, 5 or 6.

Total number of ways = ${}^6C_2=15$

The probability distribution of a random variable X is given by

X	2	3	4	5	6
P(X)	1/15	2/15	3/15	4/15	5/15

$\therefore$ Mean = $\sum XP(X)$

$$=2\times\frac{1}{15}+3\times\frac{2}{15}+4\times\frac{3}{15}+5\times\frac{4}{15}+6\times\frac{5}{15}$$

$$=\frac{2}{15}+\frac{6}{15}+\frac{12}{15}+\frac{20}{15}+\frac{30}{15}=\frac{70}{15}=\frac{14}{3}.$$

And variance = $\sum X^2P(X)-\left(\sum XP(X)\right)^2$

$$=\frac{70}{3}-\left(\frac{14}{3}\right)^2=\frac{70}{3}-\frac{196}{9}=\frac{14}{9}$$

71. Soln. Let G_i (i = 1, 2) and B_i (i = 1, 2) denote the i^{th} child is a girl or a boy respectively.

Then sample space is,

$S=\{G_1G_2, G_1B_2, B_1G_2, B_1B_2\}$

Let A be the event that both children are girls, B be the event that the youngest child is a girl and C be the event that at least one of the children is a girl.

Then A = $\{G_1G_2\}$, B = $\{G_1G_2, B_1G_2\}$

And C = $\{B_1G_2, G_1G_2, G_1B_2\}$

$\Rightarrow A\cap B=\{G_1G_2\}$ and $A\cap C=\{G_1G_2\}$

(i) Required probability = P(A/B) =

$$\frac{P(A\cap B)}{P(B)}=\frac{1/4}{2/4}=\frac{1}{2}$$

Required probability = P(A/C) =

$$\frac{P(A\cap C)}{P(C)}=\frac{1/4}{3/4}=\frac{1}{3}$$

72. Soln. Let A be the event of drawing a red ball in first draw and B be the event of drawing a red ball in second draw.

$$\therefore P(A)=\frac{{}^3C_1}{{}^{10}C_1}=\frac{3}{10}$$

Now, P(B/A) = Probability of drawing a red ball in the second draw, when a red ball already has been drawn in the first draw = $\frac{{}^2C_1}{{}^9C_1}=\frac{2}{9}$

$\therefore$ The required probability = $P(A\cap B)$

$$=P(A).P(B/A)=\frac{3}{10}\times\frac{2}{9}=\frac{1}{15}$$

12 MATERIAL

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