QUADRATIC EQUATION

* Algebraic expression

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x^1 + a_0 x^0$$

(i) $a_n \neq 0$ (ii) Power of x is whole number

is called polynomial.

eg: $x^3 - 7x^2 + \pi x + 5$ ✓, $x^7 - 6x^5 + e x^{-3} + 7$ ✗, $e x^3 - \pi x^2 + \sin x$ ✗

* $a_n \rightarrow$ leading coefficient * $a_n x^n \rightarrow$ leading term.

If leading coefficient is ~~called~~ '1' the polynomial is called as monic polynomial.

eg: $5x^6 - 7x + 8 \longrightarrow$ Monic polynomial ✗

$5x^2 - 6x + 3x + x \longrightarrow$ Monic polynomial ✓

→ Highest power of x in polynomial

Degree	Name	General Form	Example
(undefined)	Zero polynomial	$0 \cdot x^{2,3,5,\ldots}$	0
0	(Non-zero) constant polynomial	$a;(a \neq 0)$	$-2, -1, 1, 7, 3$
1	Linear polynomial	$ax + b;(a \neq 0)$	$x + 1$
2	Quadratic polynomial	$ax^2 + bx + c;\ (a \neq 0)$	$x^2 + 1$
3	Cubic polynomial	$ax^3 + bx^2 + cx + d;(a \neq 0)$	$x^3 + 1$
4	Bi-Quadratic	$ax^4 + bx^3 + cx^2 + dx + e;(a \neq 0)$	$x^4 + x^3 + 1$

$$ax^2+bx+c=0,\ (a\neq 0)$$

$$\Rightarrow\quad x^2+\frac{b}{a}x+\frac{c}{a}=0$$

$\left(\frac{\text{coeff. of } x}{2}\right)^2$ ↳ add + Subtract

$$\Rightarrow\quad x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}=0$$

$$\Rightarrow\quad \left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2}$$

$$\Rightarrow\quad x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow\quad x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow\quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$b^2-4ac=D=\text{Discriminant}$$

$$\Rightarrow\quad x=\frac{-b\pm\sqrt{D}}{2a}\ \Bigg\}\ \alpha=\frac{-b+\sqrt{D}}{2a},\quad \beta=\frac{-b-\sqrt{D}}{2a}$$

algebra

⇓

NO: of roots of a poly: of degree n is exactly n In number real or imaginary counted with multiplicity

$(x-1)^3=0$

↳ cubic

→ solution = 1 (one)

→ roots = 1,1,1 (three)

* Relation b/w roots

(1) Sum of roots

$$\alpha+\beta=\frac{-b+\sqrt{D}}{2a}+\frac{-b-\sqrt{D}}{2a}=\frac{-2b}{2a}=\frac{-b}{a}$$

$$\Rightarrow S=\alpha+\beta=\frac{-b}{a}$$

(2) Products of roots

$$P=\alpha\cdot\beta=\left(\frac{-b+\sqrt{D}}{2a}\right)\left(\frac{-b-\sqrt{D}}{2a}\right)$$

$$=\frac{b^2-D}{4a^2}=\frac{\cancel{b^2}-\cancel{b^2}+4ac}{4a^2}$$

$$\Rightarrow\quad P=\alpha\cdot\beta=\frac{c}{a}$$

β → Roots

$$P(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a\left(x^2 - (\alpha+\beta)x + \alpha\beta\right)$$

$$= a\left(x^2 - \alpha x - \beta x + \alpha\beta\right)$$

$$= a\left(x(x-\alpha) - \beta(x-\alpha)\right)$$

$$= a(x-\alpha)(x-\beta)$$

* Nature of roots

* if $D > 0 \Rightarrow$ roots are real and distinct

$$\left\{ x = \frac{-b \pm \sqrt{D}}{2a} \quad \alpha = \frac{-b+\sqrt{D}}{2a} \quad \beta = \frac{-b-\sqrt{D}}{2a} \right\}$$

* if $D = 0 \Rightarrow$ roots are real and equal

$$\left\{ \therefore \alpha = \beta = \frac{-b}{2a} \right\}$$

* if $D < 0 \Rightarrow$ Imaginary roots

Note: for real roots $D \geq 0$

* Some important points

P(1): if $a, b, c \in \mathbb{Q}$ and D is perfect square then roots are rational.

Ex: $x^2 - 2\sqrt{3}x - 1 = 0$ [if $a, b, c \notin \mathbb{Q}$]

$$x = \frac{2\sqrt{3} \pm \sqrt{16}}{2} = \sqrt{3} \pm 2$$

↳ Not Rational

* $x^2 + x + 1$

$$= (x - \omega)(x - \omega^2)$$

* $x^2 + 1 = 0$

$$x^2 = -1$$

$$x = \pm i$$

$$\therefore x^2 + 1 = (x+i)(x-i)$$

$$(\alpha-\beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta$$

$$= (\alpha+\beta)^2 - 4\alpha\beta$$

$$= \left(-\frac{b}{a}\right)^2 - \frac{4c}{a}$$

$$= \frac{b^2}{a^2} - \frac{4c}{a}$$

$$(\alpha-\beta)^2 = \frac{b^2 - 4ac}{a^2}$$

$$(\alpha-\beta)^2 = \frac{D}{a^2}$$ [Always Applicable]

$$|\alpha-\beta| = \frac{\sqrt{D}}{|a|}$$ [Not valid]

$x^3 + 1 = 0$ → $D = -4$, $a = 1$

$\alpha = i \quad \beta = -i$

$$|i - (-i)|$$

$$= |2i| = 2 \neq \frac{\sqrt{-4}}{1}$$

~~Dets~~ then roots are integers;

even + even = even
odd − even = odd

Proof: $x^2+bx+c=0$, $b, c \in I$, D is square of an integer

$$x = \frac{-b \pm \sqrt{D}}{2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

b = even integer ⇒ $x = \frac{\overset{\text{even}}{-b} \pm \overset{\text{even}}{\sqrt{b^2-4c}}}{\underbrace{2a}_{\text{even}}}$ = integer

b = odd integer $\frac{\overset{\text{odd}}{-b} \pm \overset{\text{odd}}{\sqrt{b^2-4c}}}{\underset{\text{even}}{2}}$ = integer

odd − odd = even

P(3): $a, b, c \in Q$, D is not perfect square but D > 0 then roots are irrational and occurs in conjugate pair of surds. i.e form of $p+\sqrt{q}$, $p-\sqrt{q}$

$$x = \frac{-b+\sqrt{D}}{2a}, \frac{-b-\sqrt{D}}{2a}$$

$$= \frac{-b}{2a} + \frac{\sqrt{D}}{2a}, \frac{-b}{2a} - \frac{\sqrt{D}}{2a}$$

surds
↳ Jo perfect Square नहीं ~~[illegible]~~! होते.
conjugate
↳ जिस Number से multiply करने पर Rational no: आता है

P(4): $a, b, c \in R$, $D < 0$
then ~~roo~~ imaginary roots and occurs in conjugate pairs. i.e one roots is $p+iq$ then other root is $p-iq$

Note: $a, b, c \in R$ → one root is $a+ib$ then 2nd root is $a-ib$

if $a, b, c \in Q$ → one root of quadratic is $3+\sqrt{7}$ then other root is $3-\sqrt{7}$

$x^2+x+1=0$

$$x = \frac{-1 \pm \sqrt{3}i}{2}$$

$$x = \underbrace{\frac{-1+\sqrt{3}i}{2}}_{\omega}, \underbrace{\frac{-1-\sqrt{3}i}{2}}_{\omega^2}$$

are $1, \frac{c}{a}$.

Proof

$$a(1)^2+b(1)+c=0 \Rightarrow \text{ is root:}$$

$$\therefore ax^2+bx+c=0 \begin{matrix} \nearrow 1 \\ \searrow \beta \end{matrix}$$

$$\therefore P=1.\beta=\frac{c}{a} \Rightarrow \left(\beta=\frac{c}{a}\right)$$

P(6): if $a-b+c=0$ then roots of quad. $ax^2+bx+c=0$ are $-1, -\frac{c}{a}$

Proof $\therefore$

$$ax^2+bx+c=0 \begin{matrix} \nearrow -1 \\ \searrow \beta \end{matrix}$$

$$\therefore P=(-1)(\beta)=\frac{c}{a} \Rightarrow \left(\beta=-c/a\right)$$

Equation from roots:

roots = α, β

$$\therefore eq^n = (x-\alpha)(x-\beta)=0$$

$$\Rightarrow x^2-\beta x-\alpha x+\alpha\beta=0$$

$$\Rightarrow x^2-(\alpha+\beta)x+\alpha\beta=0$$

$$\Rightarrow x^2-Sx+P=0$$

$$\left[\begin{matrix} S=\alpha+\beta \\ P=\alpha.\beta \end{matrix}\right]$$

Q: If $p(q-r)x^2+q(r-p)x+r(p-q)$ has equal roots prove $\frac{2}{q}=\frac{1}{p}+\frac{1}{r}$

M-1 $D=0$ {roots equal}

$$D=(q(r-p))^2-4p(q-r)r(p-q)=0$$

$$\Rightarrow (\underbrace{-q(p-r)}_{A+B})^2-4\underbrace{p(q-r)}_{A}\cdot\underbrace{r(p-q)}_{B}=0$$

$A=p(q-r)=pq-pr$

$B=r(p-q)=rp-qr$

$A+B=pq-qr$

$A+B=q(p-r)$

$$\Rightarrow (A+B)^2-4AB=0$$

$$\Rightarrow A^2+B^2+2AB-4AB=0$$

$$\Rightarrow (A-B)^2=0 \Rightarrow \{A=B\}$$

$\therefore p(q-r)=r(p-q)$

$pq-pr=pr-qr$

$2pr=pq+qr$

$\overline{pqr}\Big(\frac{2}{q}=\frac{1}{r}+\frac{1}{p}$ Q.E.D a b c

$\Rightarrow a+b+c = \cancel{pq} - \cancel{pr} + \cancel{qr} - \cancel{qp} + \cancel{rp} - \cancel{rq} = 0$

$\Downarrow$

roots are $1, \frac{c}{a}$

we given roots are equal

$1 = \frac{c}{a} \Rightarrow c = a \Rightarrow r(p-q) = p(q-r)$

$pr - qr = pq - pr$

$2pr = pq + qr$

$\overline{pqr}$ $\left(\left\{\frac{2}{q} = \frac{1}{r} + \frac{1}{p}\right\}\right)$

Q: If α, β are roots of $x^2-2x+5=0$ then form a quadratic equation whose roots are $\alpha^3+\alpha^2-\alpha+22$ & $\beta^3+4\beta^2-4\beta+35$

$x^2-2x+5=0 \to \alpha, \to \beta$

to form a quadratic whose roots are $\alpha^3+\alpha^2-\alpha+22$ & $\beta^3+4\beta^2-4\beta+35$

$\alpha^2-2\alpha+5=0$

$\beta^2+2\beta+5=0$

जब Higher degree से Lower degree में जाते हैं तो Division method से !!

सोच

α^3, α^2

$p(x) = \underbrace{q(x)}_{0} \cdot g(x) + r(x)$

$= r(x)$

$$\begin{array}{r|l} & \alpha+3 \\ \alpha^2-2\alpha+5 & \alpha^3+\alpha^2-\alpha+22 \\ & \underset{-}{\alpha^3}\underset{+}{-2\alpha^2}\underset{-}{+5\alpha} \\ \hline & 3\alpha^2-6\alpha+22 \\ & \underset{-}{3\alpha^2}\underset{+}{-6\alpha}\underset{-}{+15} \\ \hline & 7 \end{array}$$

$\therefore \alpha^3+\alpha^2-\alpha+22 = (\alpha^2-2\alpha+5)(\alpha+3)$

$\alpha^3+\alpha^2-\alpha+22 = +7$ (one root) $+7$

$$\begin{array}{r|l|l} \beta^2+2\beta+5 & \beta^3+4\beta^2-7\beta+35 & \beta \\ & \underset{-}{\beta^3}\underset{+}{-2\beta^2}\underset{-}{+5\beta} & \\ \hline & 6\beta^2-12\beta+35 & \\ & 6\beta^2-12\beta+30 & \\ \hline & 5 & \end{array}$$

$\therefore \beta^3-4\beta^2-7\beta+35$

$= (\beta^2+2\beta+5)(\beta+6)$

$\to \beta^3-4\beta^2-7\beta+35 = +5$ (2nd root) $+5$

$\therefore$ Eqⁿ with roots 5 & 7

$\therefore \underline{x^2-12x+35=0}$

if $f(\alpha,\beta) = F(\beta,\alpha)$ then $f(\alpha,\beta)$ is symmetric funcⁿ of roots

eg: $f(\alpha,\beta) = \alpha^2+\beta^2,\ \alpha^2\beta+\beta^2\alpha + \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

$f(\beta,\alpha) = \beta^2+\alpha^2 + \beta^2\alpha + \alpha^2\beta + \frac{\beta}{\alpha} + \frac{\alpha}{\beta}$

Note: Every symmetric function in α,β can be expressed in terms of two symmtric function.

$\alpha+\beta$ and $\alpha\cdot\beta$

NICHOD! Value of symmetric Expression in α,β can be found using sum of roots & Product of roots.

* A Golden Point:

(i) if $D_1 + D_2 \geq 0 \Rightarrow$
 → Atleast one equation has real roots.
 → if roots of one equation are imaginary then those of other equation will be real & unequal

(ii) if $D_1 + D_2 < 0 \Rightarrow$
 → Atleast one of the equation has imaginary roots.
 → if roots of one equation are real then the other equation will be Imaginary.

* Quadratic Equation v/s Identity:

If a quadratic has more than 2 distinct roots then it becomes an identity

eg) $(a+b)^2 = a^2+b^2+2ab$ (Idnty) $\sec^2\theta - \tan^2\theta = 1$ $\theta \neq (2n+1)\frac{\pi}{2}$ (Identity)

$2a+3=5$ (Eqⁿ)

Proof: $ax^2+bx+c=0 \begin{matrix}\to \alpha \\ \to \beta \\ \to \gamma\end{matrix} \Big\} \to$ distinct

$$\left.\begin{matrix} a\alpha^2+b\alpha+c=0 \\ a\beta^2+b\beta+c=0 \\ a\gamma^2+a\gamma+c=0 \end{matrix}\right\}$$

$\to a(\alpha^2-\beta^2)+b(\alpha-\beta)=0 \Rightarrow a(\alpha+\beta)+b=0$

$\to a(\beta^2-\gamma^2)+b(\beta-\gamma)=0 \Rightarrow a(\beta+\gamma)+b=0$

$a(\alpha-1)=0 \Rightarrow a=0$
$b=0$
$c=0$

Now. Quad. become

$0\cdot x^2 + 0\cdot x + 0 = 0$ (identity).

* In any polynomial equation, if the
then it becomes an identity.

* In any polynomial equation becomes an identity, then its all coefficients are simultaneously zero

↳ eg:) $\{0\cdot x^3-0\cdot x^2+0\cdot x+0=0\}$

* Newton's Formula:→

$ax^2+bx+c=0 \begin{matrix}\nearrow \alpha\\ \searrow \beta\end{matrix}$, $S_n=\alpha^n+\beta^n$, $T_n=\alpha^n-\beta^n$

$a\alpha^2+b\alpha+c=0\times\alpha^{n-2} \Rightarrow a\alpha^n+b\alpha^{n-1}+c\alpha^{n-2}=0$

$a\beta^2+b\beta+c=0\times\beta^{n-2} \Rightarrow a\beta^n+b\beta^{n-1}+c\beta^{n-2}=c$

Add $a(\alpha^n+\beta^n)+b(\alpha^{n-1}+\beta^{n-1})+c(\alpha^{n-2}+\beta^{n-2})=0$

$$\Longrightarrow \boxed{aS_n+bS_{n-1}+c\cdot S_{n-2}=0}$$

subtract: $a(\alpha^n-\beta^n)+b(\alpha^{n-1}-\beta^{n-1})+c(\alpha^{n-2}-\beta^{n-2})=0$

$$\boxed{aT_n+bT_{n-1}+cT_{n-2}=0}$$

Q $ax^2+bx+c=0 \begin{matrix}\nearrow \alpha\\ \searrow \beta\end{matrix}$ find $\alpha^4+\beta^4=??$

$aS_n+bS_{n-1}+cS_{n-2}=0$

n=4 $aS_4+bS_3+cS_2=0$ —(1)

n=3 $aS_3+bS_2+cS_1=0$

$S_1=\alpha+\beta$

$S_2=\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

⇓

S_3 — → from 1st eqⁿ We get S_4.

Q $x^2-6x-2=0 \begin{matrix}\nearrow \alpha\\ \searrow \beta\end{matrix}$, $a_n=\alpha^n-\beta^n$ then find $\frac{a_{10}-2a_8}{2a_9}$

$x^2-6x-2=0$

$\Rightarrow a_n-6a_{n-1}-2a_{n-2}=0$

$\boxed{n=10}$ $a_{10}-6a_9-2a_8=0$

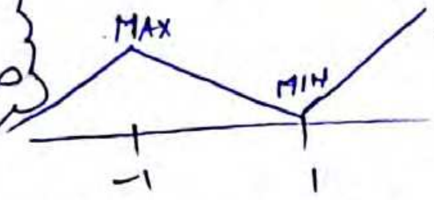
$\Rightarrow a_{10}-2a_8=3\cdot 2a_9 \Rightarrow \frac{a_{10}-2a_8}{2a_9}=③$ Ans.

Graph: $y = x + \frac{1}{x}$

$\hookrightarrow \frac{dy}{dx} = \frac{x^2-1}{x^2} = \frac{(x-1)(x+1)}{x^2}$

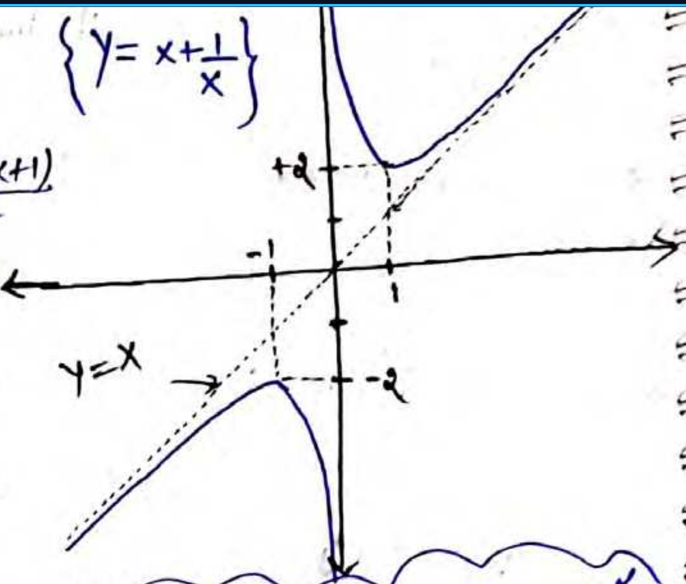
$\frac{1}{\text{very small +ve no.}} \to \infty$

$\frac{1}{\text{very small -ve no.}} \to -\infty$



$\lim_{x \to 0^+} x + \frac{1}{x} = 0 + \infty = \infty$

$\lim_{x \to 0^-} x + \frac{1}{x} = -\infty$



$a(x-\alpha)(x-\beta) = ax^2 + bx + c \to \alpha_1, \alpha_2$

$\alpha_1 + \alpha_2 = -b/a, \; \alpha_1\alpha_2 = \frac{c}{a}$

* General Polynomial Equation:-

$a_0x^3 + a_1x^2 + a_2x + a_3 = 0 \to \alpha_1, \alpha_2, \alpha_3$

$a_0x^3 + a_1x^2 + a_2x + a_3 = a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$

~~$= a_0(x^3 - x^2(\alpha_1+\alpha_2+\alpha_3) + x(\alpha_1\alpha_2 + \alpha_2\alpha_3$~~

$= a_0(x^3 - x^2(\alpha_1+\alpha_2+\alpha_3) + x(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1) - \alpha_1\alpha_2\alpha_3)$

$= a_0x^3 - a_0x^2(\alpha_1+\alpha_2+\alpha_3) + a_0x(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1) - a_0\alpha_1\alpha_2\alpha_3$

coefficient of x^2 $a_1 = -a_0(\alpha_1+\alpha_2+\alpha_3)$

coefficient of x $a_2 = a_0(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)$

constant term $a_3 = -a_0\alpha_1\alpha_2\alpha_3$

$\therefore S_1 = \alpha_1 + \alpha_2 + \alpha_3 = (-1)^1 \frac{a_1}{a_0}$

$S_2 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = (-1)^2 \frac{a_2}{a_0}$

$S_3 = \alpha_1\alpha_2\alpha_3 = (-1)^3 \frac{a_3}{a_0}$

S_1, S_3, S_5 ---- sab minus ke saath aatey hai.

S_2, S_4, S_6 ..... yeh positive ke saath.

eg: $2x^3 - 7x^2 + 6x + 5 \to \alpha, \beta, \gamma$

$S_1 = \alpha + \beta + \gamma = -(-7/2) = 7/2$

$S_2 = \alpha\beta + \beta\gamma + \gamma\alpha = 6/2 = 3$

$S_3 = \alpha_1\alpha_2\alpha_3 = -5/2.$

α_2
α_3
α_4

$$S_1 = \alpha_1+\alpha_2+\alpha_3+\alpha_4 = -\left(\frac{a_1}{a_0}\right)$$

$$S_2 = \alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_1\alpha_4+\alpha_2\alpha_3+\alpha_2\alpha_4+\alpha_3\alpha_4 = \left(\frac{a_2}{a_0}\right)$$

$$S_3 = \alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_2\alpha_3\alpha_4+\alpha_1 a_3\alpha_4 = -\left(\frac{a_3}{a_0}\right)$$

$$S_4 = \alpha_1\alpha_2\alpha_3\alpha_4 = \frac{a_4}{a_0}$$

$$\therefore a_0x^n+a_1x^{n-1}+a_2x^{n-2}+a_3x^{n-3}+\cdots\cdots+a_{n-1}x+a_n=0 \rightarrow \alpha_1, \alpha_2, \alpha_3, \vdots, \alpha_n$$

$$S_1 = \alpha_1+\alpha_2+\alpha_3+\cdots\cdots+\alpha_n = -\frac{a_1}{a_0}$$

S_2 = product taken two at a time $= \dfrac{a_2}{a_0}$

$\vdots$

$$S_n = \alpha_1\alpha_2\alpha_3\cdots\cdots\alpha_n = (-1)^n\left(\frac{a_n}{a_0}\right)$$

NOTE: Every odd degree polynomial equation with real coefficient must have at least one real root. Because imaginary roots occur in conjugate pair.

eg. $ax^3+bx^2+cx+d=0 \rightarrow 3+2i, 5, 7$ (X) $ax^3+bx^2+cx+d=0 \rightarrow 3+2i, 3-2i, 5$ (✓)

$(a,b,c,d \in R)$

eg Possible cases for 5 degree eqn
- All 5 roots real ✓
- 4 real & 1 imaginary X
- 3 real & 2 imaginary ✓
- 2 real & 3 imaginary X
- 1 real & 4 imaginary ✓
- All 5 imaginary. X

* Transformation of Equation:

$x^2-5x+6=0 \rightarrow 2=\alpha, 3=\beta$ form a quad whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}$

$\frac{1}{\alpha}, \frac{1}{\beta} \rightarrow f(\beta)=\frac{1}{\beta}$, $f(\alpha)=\frac{1}{\alpha}$

$\therefore f(x)=\frac{1}{x}$ (माना)

Put $y=f(x)=\frac{1}{x} \Rightarrow x=\frac{1}{y}$ put in original eqn.

$$\therefore \frac{1}{y^2}-\frac{5}{y}+6=0$$

$$\Rightarrow 6y^2-5y+1=0 \xleftrightarrow[y \Longleftrightarrow x]{\text{Replace}} 6x^2-5x+1$$

$$a_1x+b_1y+c_1=0 \times b_2$$
$$a_2x+b_2y+c_2=0 \times b_1$$
$$- \quad - \quad -$$
$$(a_1b_2-b_1a_2)x+b_2c_1-b_1c_2=0$$

$$x=\frac{b_1c_2-b_2c_1}{a_1b_2-b_1a_2}=\frac{\begin{vmatrix}b_1 & c_1\\ b_2 & c_2\end{vmatrix}}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}}$$

$$a_1x+b_1y+c_1=0\times a_2$$
$$a_2x+b_2y+c_2=0\times a_1$$
$$- \quad - \quad -$$
$$(b_1a_2-a_1b_2)y+a_2c_1-a_1c_2=0$$

$$y=\frac{a_2c_1-a_1c_2}{a_1b_2-b_1a_2}=\frac{\begin{vmatrix}c_1 & a_1\\ c_2 & a_2\end{vmatrix}}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}}$$

Finally if $a_1x+b_1y+c_1=0$ & $a_2x+b_2y+c_2=0$

Then
$$x=\frac{\begin{vmatrix}b_1 & c_1\\ b_2 & c_2\end{vmatrix}}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}}, \quad y=\frac{\begin{vmatrix}c_1 & a_1\\ c_2 & a_2\end{vmatrix}}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}}$$

* condition for common roots

$$\left.\begin{array}{l} a_1x^2+b_1x+c_1=0\\ a_2x^2+b_2x+c_2=0\end{array}\right\} \rightarrow \text{Have a common root } \alpha$$

$$\therefore \left.\begin{array}{l} a_1\alpha^2+b_1\alpha+c_1=0\\ a_2\alpha^2+b_2\alpha+c_2=0\end{array}\right\} \quad \alpha^2=\frac{\begin{vmatrix}b_1 & c_1\\ b_2 & c_2\end{vmatrix}}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}}, \quad \alpha=\frac{\begin{vmatrix}c_1 & a_1\\ c_2 & a_2\end{vmatrix}}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}}$$

$$\therefore \frac{\begin{vmatrix}c_1 & a_1\\ c_2 & a_2\end{vmatrix}^2}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}^{\cancel{2}}}=\frac{\begin{vmatrix}b & c_1\\ b_2 & c_2\end{vmatrix}}{\cancel{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}}} \Rightarrow \begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}\cdot\begin{vmatrix}b_1 & c_1\\ b_2 & c_2\end{vmatrix}=\begin{vmatrix}c_1 & a_1\\ c_2 & a_2\end{vmatrix}^2$$

Finally

if $a_1x^2+b_1x+c_1=0$
$a_2x^2+b_2x+c_2=0$ have a common root α

then
$$\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}\cdot\begin{vmatrix}b_1 & c_1\\ b_2 & c_2\end{vmatrix}=\begin{vmatrix}c_1 & a_1\\ c_2 & a_2\end{vmatrix}^2$$

↳ This is condition for atleast one root common root.

$\therefore\ a_1x^2+b_1x+c_1=0 \begin{smallmatrix}\nearrow \alpha \\ \searrow \beta\end{smallmatrix} \rightarrow S = \alpha+\beta$

$a_2x^2+b_2x+c_2=0 \begin{smallmatrix}\nearrow \alpha \\ \searrow \beta\end{smallmatrix} \rightarrow S = \alpha+\beta = \frac{-b_2}{a_2} \quad P = \alpha\cdot\beta = \frac{c_2}{a_2}$

Hence condition for Both roots common.

$\Downarrow$

$+\frac{b_1}{a_1} = +\frac{b_2}{a_2} \Big\} \ \frac{c_1}{a_1} = \frac{c_2}{a_2}$

$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$\rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} \Big\} \ \frac{a_1}{a_2} = \frac{c_1}{c_2}$

NOTE:

$a_1x^2+b_1x+c_1=0 \begin{smallmatrix}\nearrow p+iq \\ \searrow p-iq\end{smallmatrix}$

$a_2x^2+b_2x+c_2=0 \begin{smallmatrix}\nearrow p+iq \\ \searrow p-iq\end{smallmatrix}$

Have a common roots & the roots of one of the equations are Imaginary then both roots will be common

if the equation

$F(x)=0 \rightarrow \alpha$
$g(x)=0 \rightarrow \alpha$ } Have a common root say α then $x=\alpha$ is also $Af(x) \pm Bg(x) = 0 \rightarrow \alpha$

eg: $x^2-5x+6=0 \begin{smallmatrix}\nearrow 2 \\ \searrow 3\end{smallmatrix}$

$x^2-6x+8=0 \begin{smallmatrix}\nearrow 4 \\ \searrow ②\end{smallmatrix}$

$\underline{-\quad +\quad -}$

$x-2=0 \Rightarrow \boxed{x=2}$

Q ~~let α,β~~, $x^3+2x^2-4x+5=0 \begin{smallmatrix}\nearrow \alpha \\ \rightarrow \beta \\ \searrow \gamma\end{smallmatrix}$. find $\frac{(\alpha^3+5)(\beta^3+5)(\gamma^3+5)}{13\alpha\cdot\beta\cdot\gamma}$

$\alpha^3+5 = 4\alpha-2\alpha^2 = 2\alpha(2-\alpha)$

$\alpha^3+5 = 2\alpha(2-\alpha)$

Illy $\beta^3+5 = 2\beta(2-\beta)$

$\gamma^3+5 = 2\gamma(2-\gamma)$

$E = \frac{(\alpha^3+5)(\beta^3+5)(\gamma^3+5)}{13\alpha\cdot\beta\cdot\gamma}$

$= \frac{2\cancel{\alpha}(2-\alpha)\cdot 2\cancel{\beta}(2-\beta)\cdot 2\cancel{\gamma}(2-\gamma)}{13\cancel{\alpha}\cdot\cancel{\beta}\cdot\cancel{\gamma}}$

put eqn (1)

$\Rightarrow \frac{8(2-\alpha)(2-\beta)(2-\gamma)}{13}$

$\Rightarrow \frac{8\times 13}{13} = \boxed{8}$

MAST TARIKA

$x^3+2x^2-4x+5 = 1\cdot(x-\alpha)(x-\beta)(x-\gamma)$

$\alpha \quad \beta \quad \gamma$

put $x=2$

$8+8-8+5 = (2-\alpha)(2-\beta)(2-\gamma)$

$\Rightarrow (2-\alpha)(2-\beta)(2-\gamma) = 13$ —(1)

$$f(x) = ax^2+bx+c = a(x-\alpha)(x-\beta)$$

$$\text{if } D=0 \Rightarrow \boxed{\alpha=\beta}$$

$$\therefore f(x) = a(x-\alpha)(x-\alpha)$$

$$f(x) = a(x-\alpha)^2$$

$a = +ve$

$$f(x) = (\sqrt{a})^2(x-\alpha)^2$$
$$= [\sqrt{a}(x-\alpha)]^2$$
$$\Downarrow$$

f(x) becomes square of a linear polynomial with real coeff.

$a = -ve$

$$f(x) = (\sqrt{|a|}\,i)^2(x-\alpha)^2$$
$$= [(\sqrt{|a|}\,i)(x-\alpha)]^2$$

$$(\sqrt{|-4|}\,i)$$
$$-4 = (2i)^2$$
$$-16 = (4i)^2$$
$$\Downarrow$$
$$(\sqrt{|-16|}\,i)^2$$
$$\Downarrow$$
$$(4i)^2$$

Q $f(x) = \lambda x^2 - 2\lambda x + 1$

Find λ if (a) $f(x)$ is square of a linear/perfect square

Solⁿ

$$D=0$$
$$4\lambda^2 - 4\lambda = 0 \quad \& \; a>0 \quad \therefore \underline{\lambda>0}$$
$$4\lambda(\lambda-1) = 0$$
$$\lambda = \underset{\otimes}{0},\; \lambda = \underline{1}\,✓$$

$f(x) = \lambda x^2 - \lambda x - 4$

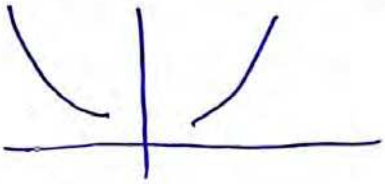
Find λ if $f(x)$ is square of linear with real coeff.

Solⁿ $D=0$

$$\lambda^2 + 16\lambda = \lambda(\lambda+16) = 0$$
$$\lambda = \underset{\otimes}{0},\; \underset{\otimes}{-16}$$

ANS: No solⁿ value of λ exist!

* Analysis of a Quadratic Polynomial:



$$y = f(x) = ax^2+bx+c, \quad (a\neq 0)$$
$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c$$
$$\Rightarrow a\left(x+\frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}$$
$$f(x) = y = a\left(x+\frac{b}{2a}\right)^2 - \frac{D}{4a}$$

$$f\left(\frac{-b}{2a}+k\right) = a\left(\frac{-b}{2a}+k+\frac{b}{2a}\right)^2 - \frac{D}{4a} = ak^2 - \frac{D}{4a}$$
$$f\left(\frac{-b}{2a}-k\right) = a\left(\frac{-b}{2a}-k+\frac{b}{2a}\right)^2 - \frac{D}{4a} = ak^2 - \frac{D}{4a}$$

Same.

$$y = f(x) = ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 - \frac{D}{4a}$$

if $a>0$

$y_{max} \rightarrow \infty$

$y_{min} \rightarrow \frac{-D}{4a}$ at $x = -\frac{b}{2a}$

If $a<0$

$y_{min} \rightarrow -\infty$

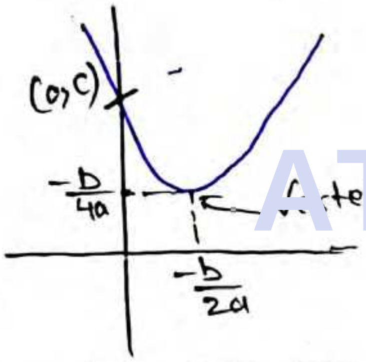
$y_{max} = \frac{-D}{4a}$ at $\frac{-b}{2a}$.

$a>0$

$$\underbrace{y = ax^2+bx+c}$$

$y_{max} \rightarrow \infty$

$y_{min} = \frac{-D}{4a}$ at $\frac{-b}{2a}$.

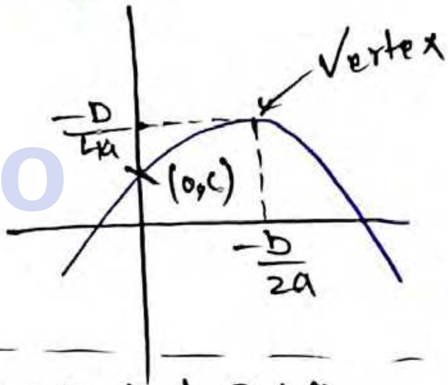


$a<0$

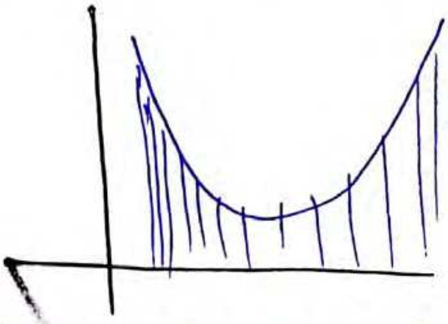
$$\underbrace{y = ax^2+bx+c}$$

$y_{max} = \frac{-D}{4a}$ at $x = -\frac{b}{2a}$

$y_{min} \rightarrow -\infty$



If $a>0$ but $D<0$ +ve



$y_{min} = \frac{-D}{4a}$

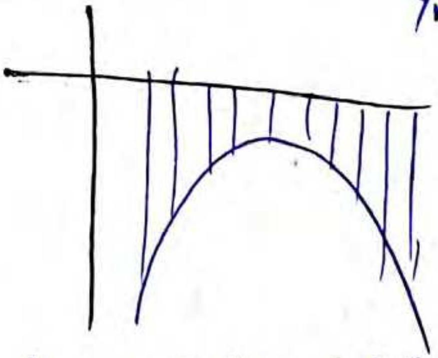
$y_{max} \rightarrow \infty$

then $ax^2+bx+c>0$

$\forall\ x \in R$

If $a<0$ but $D<0$ -ve

$y_{max} = \frac{-D}{4a}$



$y_{min} = \infty$

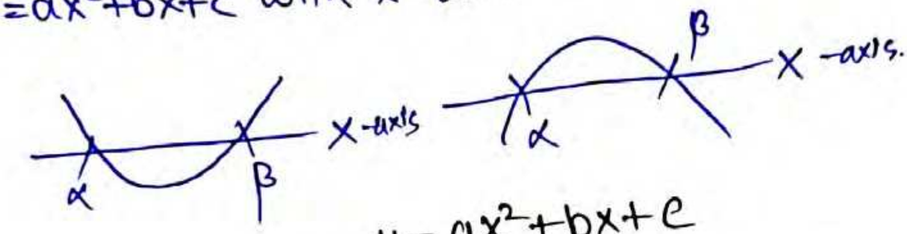
then $ax^2+bx+c<0$

$\forall\ x \in R$

Quadratic Equation:

$y = ax^2 + bx + c \quad (a \neq 0)$

if we put $y = 0$ we get $ax^2+bx+c=0$ called Quadratic equation and roots are the nothing but x-coordinates of the point of Intersection of $y=ax^2+bx+c$ with x-axis.



$y = ax^2+bx+c$

If $D=0$

$a>0$

$y_{MAX} \to \infty$

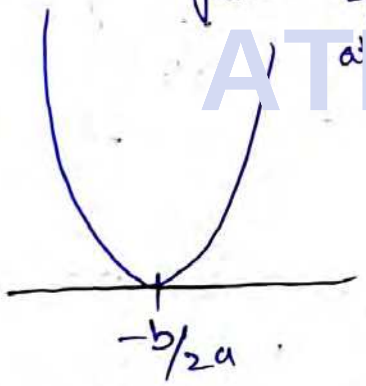
$y_{min} = \frac{-D}{4a} = 0$

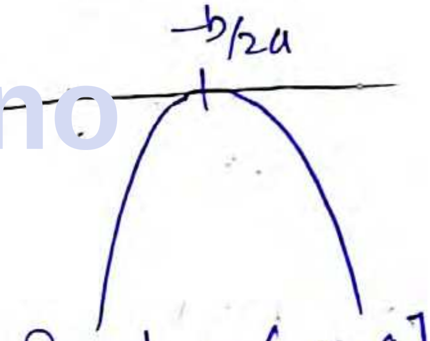
at $x =$

$a<0$

$y_{MAX} = \frac{-D}{4a} = 0$ at $\frac{-b}{2a}$

$y_{MIN} \to -\infty$





Range of $y = (-\infty, 0]$

Range of $y = [0, \infty)$

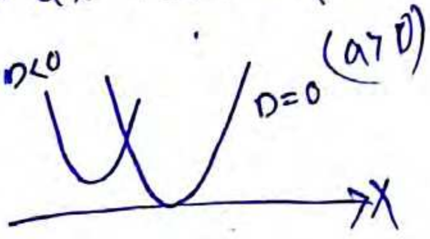
* Important:

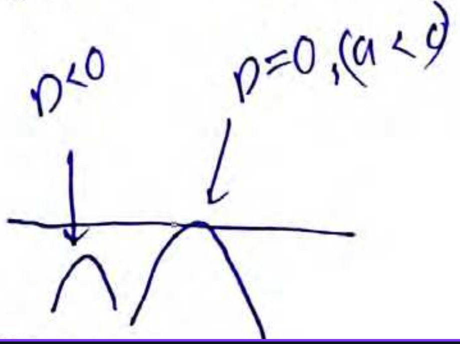
→ if $a>0$ and $D<0$ then $y = ax^2+bx+C > 0 \ \forall \ x \in R$

→ if $a<0$ and $D<0$ then $y = ax^2+bx+e < 0 \ \forall \ x \in R$

→ $ax^2+bx+e \geq 0 \ \forall \ x \in R$ then $a>0$ & $D \leq 0$

→ $ax^2+bx+C \leq 0 \ \forall \ x \in R$ then $a<0$ & $D \leq 0$





Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$, then find the value of $P(4)$.
$P(0)=6$, $P(1)=7$, $P(2)=8$ and $P(3)=9$, then find the value of $P(4)$.

Pattern को 4'

<u>let</u> $g(x) = P(x) - (x+6) \rightarrow$ polynomial of degree 4

Pattern को पहचान कर Equation form करो!!

$$\left.\begin{aligned} g(0) &= P(0)-6=0 \\ g(1) &= P(1)-7=0 \\ g(2) &= P(2)-8=0 \\ g(3) &= P(3)-9=0 \end{aligned}\right\} \rightarrow g(0)=g(1)=g(2)=g(3)$$

$$\Downarrow$$

$$g(x) = (x-0)(x-1)(x-2)(x-3)$$

$$P(x)-(x+6) = x(x-1)(x-2)(x-3)$$

$$\Rightarrow P(x) = x(x\cancel{+}1)(x-2)(x-3) + (x+6)$$

$$\Rightarrow P(4) = 4\cdot3\cdot2\cdot1 + 10 = \boxed{34}$$

* Range of Rational functions :-

Type(1) : $f(x) = \dfrac{\text{linear}_1}{\text{linear}_2}$ i.e $f(x) = \dfrac{ax+b}{cx+d}$ $\left(\dfrac{a}{c} \neq \dfrac{b}{d}\right)$

if $\dfrac{a}{c} = \dfrac{b}{d}$

$f(x) = \dfrac{3x+2}{6x+4}$ $\left(\dfrac{3}{6}=\right.$

$= \dfrac{\cancel{3x+2}}{2\cancel{(3x+2)}} = 1/2$

$$y = \frac{ax+b}{cx+d} \Rightarrow ycx + yd = ax + b$$

$$\Rightarrow x(cy-a) = b-yd$$

$$\Rightarrow x = \frac{b-yd}{cy-a} \Rightarrow cy-a \neq 0 \Rightarrow y \neq a/c$$

Hence $\Rightarrow$ $y \in R - \left\{\dfrac{a}{c}\right\}$

eg. $f(x) = \dfrac{2x-3}{x-1}$ $\Rightarrow R_f = R - \{2\}$

Type(2) : $f(x) = \dfrac{\text{quad}_1}{\text{quad}_2}$ where numerator & denominator contain a common factor

<u>Steps:</u>

(i) Factorize numerator & Denominator

(ii) Say $(ax-b)$ is a common factor, cancel $(ax-b)$ from numerator & denominator & write $x = b/a$

(iii) Now $f(x) = \dfrac{px+q}{rx+s}$, $x \neq b/a$ & range $= R - \left\{\dfrac{p}{r}, f\left(\dfrac{b}{a}\right)\right\}$

eg: $f(x) = \dfrac{x^2-5x+6}{x^2-6x+8} \Rightarrow f(x) = \dfrac{(x-3)\cancel{(x-2)}}{\cancel{(x-2)}(x-4)}$ $(x \neq 2)$

$f(x) = \dfrac{x-3}{x-4}$, $(x \neq 2)$

Range : $R - \{1, 1/2\}$

if $x=2$, $f(x) = \dfrac{2-3}{2-4} = 1/2$

* **Type (3)**: $f(x) = \frac{\text{Quad}}{\text{Quad}}, \frac{\text{linear}}{\text{Quad}}, \frac{\text{Quad}}{\text{linear}}$, where numer
denominator contain no common Factor.

Steps:

(i) Equate given expression to y
(ii) Cross multiply & make quad in x
(iii) Since $x \in R$, put $D \geq 0$ & get range of y
(iv) Equate coeff of $x^2 = 0$ to get $y = y_1$ put expression $= y_1$ & solve for x

If we get real x answer of step 3 is final

if we do not get real x exclude y_1 from answer.

Q Find values of 'a' for which the expression $\frac{ax^2+3x-4}{3x-4x^2+a}$ assumes all real values for real values of x.

given: Range = R, a = ?
$y \in R$

$$y = \frac{ax^2+3x-4}{3x-4x^2+a}$$

$$\Rightarrow y = \frac{ax^2+3x-4}{-4x^2+3x+a}$$

For range to be R: Numerator & denominator should not have a common root

for common root $\begin{vmatrix} a & 3 \\ -4 & 3 \end{vmatrix} \cdot \begin{vmatrix} 3 & -4 \\ 3 & a \end{vmatrix} = \begin{vmatrix} -4 & a \\ a & -4 \end{vmatrix}^2$

$$\Rightarrow (3a+12)^2 = (16-a^2)^2$$

$$\Rightarrow (3a+12)^2 - (16-a)^2 = 0$$

$$\Rightarrow (3a+12+16-a^2)(3a+12-16+a^2) = 0$$

$$\Rightarrow (-a^2+3a+28)(a^2+3a-4) = 0$$

$$\Rightarrow (a^2-3a-28)(a^2+3a-4) = 0$$

$$\Rightarrow a^2-3a-28 = 0 \quad , \quad a^2+3a-4 = 0$$

$a = 7, -4$ $\quad$ $a = -4, 1$

$$a \neq -4, 7, 1$$

$$3x-4x^2+a$$

$$\Rightarrow 3xy-4yx^2+ay=ax^2+3x-4$$

$$\Rightarrow (a+4y)x^2+(3-3y)x-4-ay=0$$

Since $x\in R$

$\underline{D\geq 0,}$ $9(1-y)^2+4(a+4y)(4+ay)\geq 0$

$$\Rightarrow 9y^2+9-18y+4(4a+a^2y+16y+4ay^2)\geq 0$$

$$\Rightarrow y^2(16a+9)+y(4a^2+46)+9+16a\geq 0 \ \forall\ y\in R$$

$\boxed{D\leq 0,\ 16a+9>0}$ $(4a^2+46)^2-4\cdot(16a+9)(16a+9)\leq 0$, $\boxed{a>-\frac{9}{16}}$ —(i)

$$\Rightarrow (4a^2+46)^2-(2(16a+9)^2\leq 0$$

$$\Rightarrow (4a^2+46-32a-18)(4a^2+46+32a+18)\leq 0$$

$$\Rightarrow (4a^2-32a+28)(4a^2+32a+64)\leq 0$$

$$\Rightarrow (a^2-8a+7)(a^2+8a+16)\leq 0$$

$$\Rightarrow (a-1)(a-7)(a+4)^2\leq 0$$

$\Rightarrow (a-1)(a-7)\leq 0$, $a=-4$ is also possible.

$\Rightarrow a\in[1,7]\cup\{-4\}$ —(ii)

From eqⁿ (i) and (ii) $a\in[1,7]$ But $\boxed{a\neq 7,1}$

$\therefore$ $\underline{\text{ANS: } a\in(1,7)}$

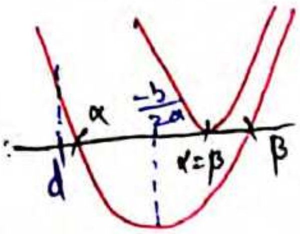
* Location of roots:→

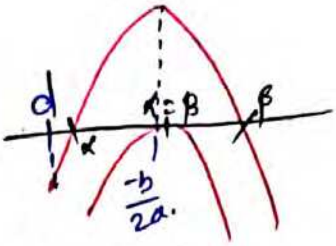
The article deals with an elegant approach of solving problems on quadratic equations when the roots are located/specified on the number line with variety of constraints:

consider: $f(x)=ax^2+bx+c$

Type 1 : Both roots greater than

$f(x) = ax^2 + bx + C$





(1) $D \geq 0$

(2) $\frac{-b}{2a} > d$

(3) $f(d) > 0$

(4) $a > 0$

(1) $D \geq 0$

(2) $\frac{-b}{2a} > d$

(3) $f(d) < 0$

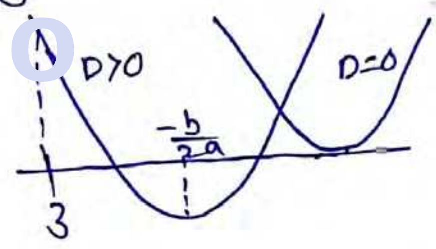
(4) $a < 0$

combine

(a) $D \geq 0$ (b) $\frac{-b}{2a} > d$ (c) $af(d) > 0$

eg: Find all the values of the parameter 'd' for which both the roots of the eq^n $x^2 - 6dx + (2-2d-9d^2) = 0$ exceed the number 3

$f(x) = x^2 - 6dx + (2-2d-9d^2) = 0$



(1) $D = 36d^2 - 4 \cdot (2-2d+9d^2) \geq 0$

$\cancel{9d^2} - 2 + 2d - \cancel{9d^2} \geq 0$

$2d - 2 \geq 0$

$\boxed{d \geq 1}$ — I

(1) $D \geq 0$

(2) $f(3) > 0$

(3) $\frac{-b}{2a} > 3$

(2) $f(3) > 0$

$9 - 18d + 2 - 2d + 9d^2 > 0$

$9d^2 - 20d + 11 > 0$

$9d^2 - 11d - 9d + 11 > 0$

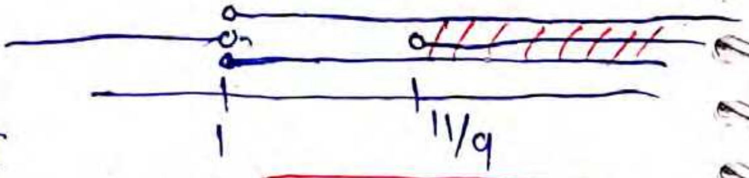
$(9d-11)(d-1) > 0$

$\boxed{d \in (-\infty, 1) \cup (11/9, \infty)}$ — II

(3) $\frac{-b}{2a} > 3$

$\Rightarrow \frac{6d}{2} > 3 \Rightarrow 3d > 3$

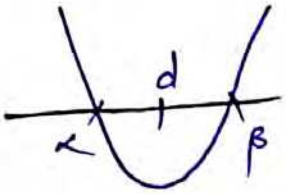
$\Rightarrow \boxed{d > 1}$ — III



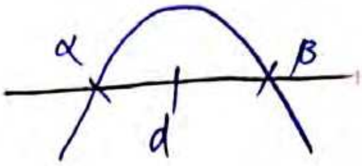
ANS : $\boxed{d \in (11/9, \infty)}$

Type 2: Both roots

b/w roots or one root is less than 'd' & other root is greater than 'd'

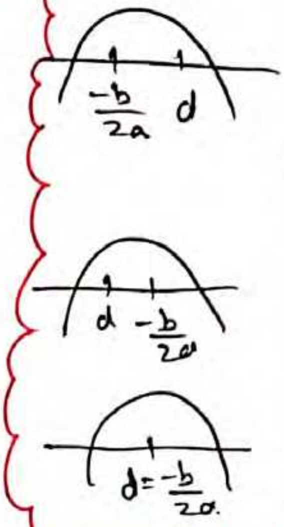


(i) $a > 0$
(ii) $D > 0$
(iii) $f(d) < 0$



(i) $a < 0$
(ii) $f(d) > 0$
(iii) $D > 0$

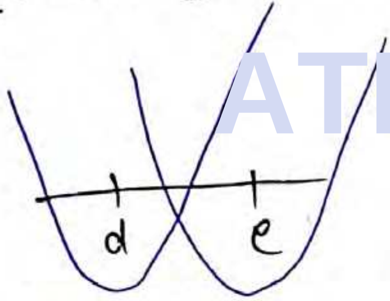
can not comment on $= \frac{-b}{2a}$



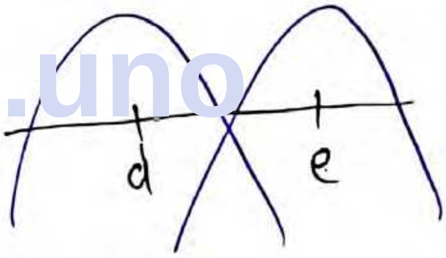
combine

(i) $a \cdot f(d) < 0$
(ii) $D > 0$ (No Need)

Type 3: Exactly one root lies b/w d & e (d<e)



(i) $f(d) \cdot f(e) < 0$
(ii) $a > 0$
(iii) $D > 0$



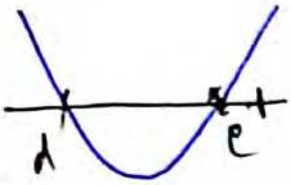
(i) $f(d) \cdot f(e) < 0$
(ii) $a < 0$
(iii) $D > 0$

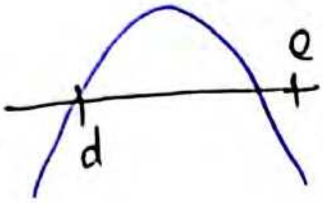
combine

(i) $f(d) \cdot f(e) < 0$
(ii) $a \neq 0$
(iii) $D > 0$ (No Need)

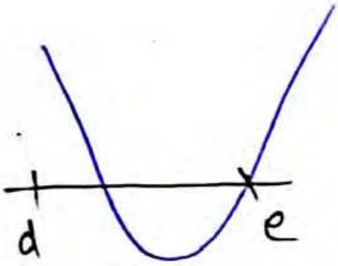
Next

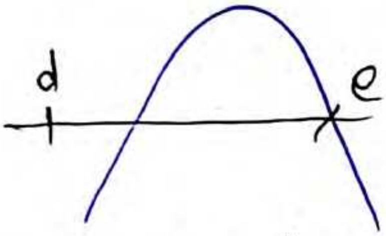
Now possibilities arise :





One root lies at $x = d$ i.e $f(d) = 0$, then find other root it must lie in (d, e)



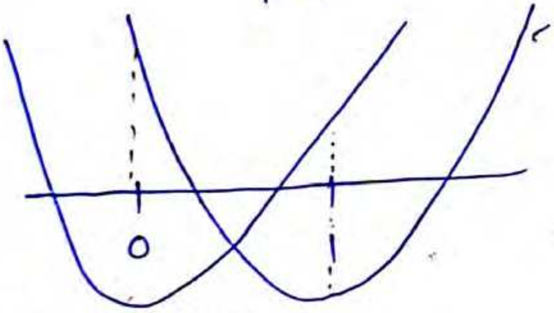


One root lies at $x = e$ i.e $f(e) = 0$ then find the other root it must lie in (d, e)

Question : $(\lambda^2+1)x^2 - 4\lambda x + 2 = 0$ always have exactly one root in the interval $(0,1)$ is: find set of 'λ'

exactly one root lies in $(0,1)$

$$y = \underbrace{(\lambda^2+1)}_{+ve} x^2 - 4\lambda x + 2 = 0$$



$f(0) \cdot f(1) < 0$

$\Rightarrow 2 \cdot (\lambda^2 + 1 - 4\lambda + 2) < 0$

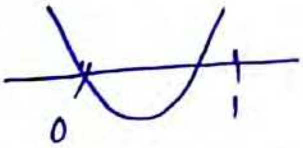
$\Rightarrow (\lambda^2 - 4\lambda + 3) < 0$

$\Rightarrow (\lambda - 3)(\lambda - 1) < 0$

$\Rightarrow \lambda \in (1, 3)$ —— (i)

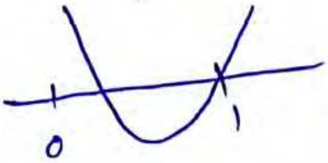
Now possibility arise

Ⓟ₁



$f(0) = 0 \Rightarrow 2 = 0$
↓
Not possible

Ⓟ₂



$f(1) = 0 \Rightarrow \lambda^2 + 1 - 4\lambda + 2 = 0$

$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$

$\underline{\lambda = 1, 3}$

$2x^2-4x+2=0$

$x^2-2x+1=0$

$x=1,1$

$\Rightarrow \lambda=1$ is rejected

$10x^2-12x+2=0 \rightarrow \alpha$

M-1

$5x^2-6x+1=0$

$(5x-1)(x-1)=0$

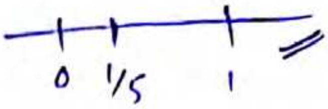
$x=1/5, 1$

M-2 (easier method)

$\alpha \cdot 1 = 2/10$

$\boxed{x=1/5}$

Hence $\lambda=3$ is accepted



$$\boxed{\therefore \lambda \in (1,3) \cup \{3\} \Rightarrow \lambda \in (1,3]}$$

2. IIT can modify the question as!!

Not $(0,3)$

Q $x^2-(a+1)x+2a=0$ has exactly one root in $[0,3]$ find range of a

$f(x)=x^2-(a+1)x+2a=0$

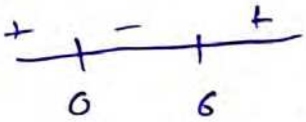
$f(0) \cdot f(3) < 0$

$\Downarrow$

$\Rightarrow$ [illegible] $) < 0$

$\Rightarrow a(6-a) < 0$

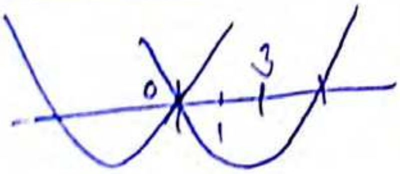
$\Rightarrow a(a-6) > 0$



$a \in (-\infty,0) \cup (6,\infty)$ —(i)

Now two possibilities arise

(P₁)



$f(0)=0 \Rightarrow a=0$

eqⁿ → $x^2-x=0$

$x=0,1$

$a \neq 0$ rejected ~~rejected~~

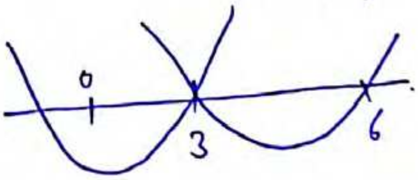
bcoz at $a=0$

$x=0,1$ ~~ka~~ and $[0,3]$

~~has~~ x have 2 roots

but ACTQ only one root lie b/w $[0,3]$

(P₂)



$f(3)=0 \Rightarrow 6-a=0 \Rightarrow \boxed{a=6}$

eqⁿ $x^2-7x+12=0 \rightarrow 3, \beta$

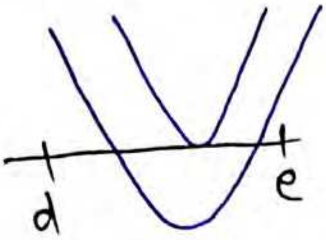
$3 \cdot \beta = 12$

$\boxed{\beta=4}$

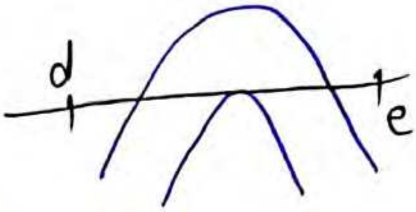
($a=6$ is accepted) —(ii)

$\therefore a \in (-\infty,0) \cup (6,\infty) \cup \{6\}$

$$\Rightarrow \boxed{a \in (-\infty,0) \cup [6,\infty)}$$



OR



(1) $f(d) > 0$

(2) $f(e) > 0$

(3) $D \geq 0$

(4) $d < \frac{-b}{2a} < e$

(5) $a > 0$

(1) $f(d) < 0$

(2) $f(e) < 0$

(3) $D \geq 0$

(4) $d < \frac{-b}{2a} < e$

(5) $a < 0$

combine

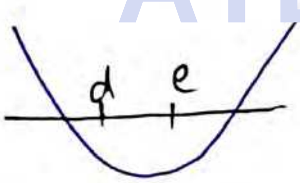
(i) $D \geq 0$

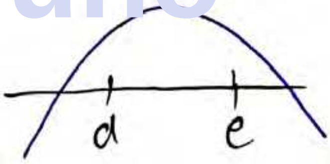
(ii) $d < \frac{-b}{2a} < e$

(iii) $a \cdot f(d) > 0$

(iv) $a \cdot f(e) > 0$

Type V : One root is greater than e & other root is less than d (d<e





(1) $a > 0$

(2) $f(d) < 0$

(3) $f(e) < 0$

(4) $D > 0$

(1) $a < 0$

(2) $f(d) > 0$

(3) $f(e) > 0$

(4) $D > 0$

combine

(i) $a \cdot f(d) < 0$

(ii) $a \cdot f(e) < 0$

(iii) $D > 0$ (No Need)

$ax^2+bx+c=0$, where $a, b, c \in Q$ has rational roots if D is a perfect Square

Q Find no: of integral values α for which the quadratic eqn $x^2+\alpha x+\alpha+1=0$ has integral roots.

$x^2+\alpha x+\alpha+1=0$ we can assume α as integer clearly

all coeff are rational

$\therefore$ for roots be rational $\Rightarrow$ D should be perfect square.

$D = \alpha^2-4(\alpha+1) = \alpha^2-4\alpha-4 = m^2$, $m \in N$

$\Rightarrow \alpha^2-4\alpha-4 = m^2$

$\Rightarrow (\alpha-2)^2-8 = m^2$

$\Rightarrow (\alpha-2)^2-m^2 = 8$

$\Rightarrow \overset{A}{(\alpha-2-m)}\overset{B}{(\alpha-2+m)} = 8 \Rightarrow 1\times8,\ 2\times4,\ -8\times-1,\ -4\times-2$

m plus होता है तो B ~~A~~ A से बड़ा आएगा

इसलिए [illegible] cases की जरूरत नहीं

1×8 ✓ 8×1 ⊗

$\alpha-2-m=1$
$\alpha-2+m=8$
$2\alpha-4=9$
$\boxed{\alpha=13/2}$ ✗
(bcoz not an integer)

$\alpha-2-m=2$
$\alpha-2+m=4$
$2\alpha-4=6$
$\boxed{\alpha=5}$ ✓

$\alpha-2-m=-8$
$\alpha-2+m=-1$
$2\alpha-4=-9$
$\boxed{\alpha=-5/2}$ ✗
(bcoz not an integer)

$\alpha-2-m=-4$
$\alpha-2+m=-2$
$2\alpha-4=-6$
$\boxed{\alpha=-1}$ ✓

$\Rightarrow$ if $\alpha=-1$ or 5 root are rational

if $\alpha=-1$ eqn becomes $x^2-x=0 \Rightarrow x=0,1$
if $\alpha=5$ eqn becomes $x^2+5x+6=0 \Rightarrow x=-2,-3$
Both roots integer

Hence $\alpha=-1$ & $\alpha=5$ is valid for integral roots.

M-2

$x^2 + \alpha x + \alpha + 1 = 0 \begin{matrix} \nearrow a \\ \searrow b \end{matrix}$ } integral

$S = a + b = -\alpha$

$P = ab = \alpha + 1$

$a + b + ab = 1$ { Simon's Factoring Technique }

$\Rightarrow (a+1) + b(a+1) - 1 = 0$

$\Rightarrow (a+1)(b+1) = 2$

$\Rightarrow a + 1 = 1$, $b + 1 = 2$ ⟹ $a = 0, b = 1$ OR $a + 1 = -2$, $b + 1 = -2$ ⟹ $a = -2, b = -3$ $\boxed{\alpha = -1, 5}$ //

$\Rightarrow -\alpha = a + b$ $\boxed{\alpha = -1}$ $-\alpha = -5$ ∴ $\boxed{\alpha = 5}$

* condition for general two degree expression in x & y to be resolved as a product of two linear factors.

$$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

$$= ax^2 + (2hy + 2g)x + by^2 + 2fy + c$$

{ let us find its roots. } ∴

$$x = \frac{-(2hy+2g) \pm \sqrt{(2hy+2g)^2 - 4 \cdot a \cdot (by^2 + 2fy + c)}}{2a}$$

$$= \frac{-(hy+g) \pm \sqrt{(h^2 - ab)y^2 + (2hg - 2af)y + g^2 - ac}}{a}$$

$$x = \frac{-(hy+g) + \sqrt{(h^2 - ab)y^2 + (2hg - 2af)y + g^2 - ac}}{a} = \alpha$$

$$x = \frac{-(hy+g) - \sqrt{(h^2 - ab)y^2 + (2hg - 2af)y + g^2 - ac}}{a} = \beta$$

∴ for linear factors

$px+qy+\gamma$
↓
linear in x&y

$(h^2-ab)y^2-y(2hg-2af)+g^2-ac$ should become a perfect square

$\Rightarrow \{D=0\}$

$D=(2hg-2af)^2-4(h^2-ab)(g^2-ac)=0$

$\frac{(\;)}{4}\Big(\Rightarrow 4(hg-af)^2-(h^2-ab)(g^2-ac)=0$

$\Rightarrow h^2g^2+a^2f^2-2afhg-[h^2g^2-h^2ac-g^2ab+a^2bc]=0$

$\frac{(\;)}{a}\Big(\; af^2-2fgh+ch^2+bg^2-abc=0$

$\Rightarrow \boxed{abc+2fgh-af^2-bg^2-ch^2=0}$

$f(x,y)=ax^2+2hxy+by^2+2gx+2fy+c$ can be resolved in to two linear factors if

$\Delta = abc+2fgh-af^2-bg^2-ch^2=0$

OR

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Q: $2x^2+3xy+y^2+3x+1+2y$ Factorize into linear factors

$f(x,y)=2x^2+3xy+y^2+3x+1+2y$

$a=2$, $b=1$, $2h=3$, $2g=3$, $2f=2$, $c=1$
$h=3/2$, $g=3/2$, $f=1$

$\Delta = abc+2fgh-af^2-bg^2-ch^2$

$= 2\cdot1\cdot1+2\cdot1\cdot\frac{3}{2}\cdot\frac{3}{2}-2\cdot1^2-1(3/2)^2-1\cdot(3/2)^2$

$= 2+9/2-2-9/4-9/4$

$=0$

∴ given Expression can be factorized into 2 linear factor.

$F(x,y) = 2x + 3xy + y$

Steps to Factorize

* Factorize the Homogenous part (degree same)

$$2x^2+3xy+y^2 = 2x^2+2xy+xy+y^2$$
$$= 2x(x+y)+y(x+y)$$
$$=(2x+y)(x+y)$$

* Add λ to First factor & μ to second factor & put their product equal to given expression.

$$(2x+y+\lambda)(x+y+\mu) = 2x^2+3xy+y^2+3x+2y+1$$

coefficient of x : $2\mu+\lambda = 3$

coefficient of y : $\mu+\lambda = 2$

$\mu = 1, \lambda = 1$

$2\mu x+\lambda x$
$\underbrace{(2\mu+\lambda)}_{\text{coeff. of } x}x$

$\mu y+\lambda y$
$\underbrace{(\mu+\lambda)}_{\text{coeff of } y}y$

$$\boxed{\therefore\ 2x^2+3xy+y^2+3x+2y+1 = (2x+y+1)(x+y+1)}$$

* A Golden Point :

for $y = f(x) = ax^2+bx+c$ $(a \neq 0)$
if $f(p)<0$ and $f(q)>0$
i.e $f(p)\cdot f(q)<0 \Rightarrow$ then the equation $ax^2+bx+c=0$ has exactly one root lying b/w p & q.