SEQUENCE & Series

* **Sequence:** A succession of terms which may be algebraic, real or complex number, written one after other separated by commas is called sequence. It is represented by $<a_n>$ or $\{a_n\}$

eg: 2, 3, 5, 7, 11, 18 . . . .

−1, 1, −1, 1, ...

> Every Progression is a sequence but every sequence is not a progression

* **Progression:** Speacial case of sequence in which the terms progress according to a definite rule.

eg: $<n^3-1> \equiv 0, 7, 26, \ldots$ (T_1 = Term 1, Term 2 = T_2, T_3)

eg: $\left\{\frac{n}{n^2+1}\right\} \equiv \frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \ldots$

* **Series:** if we add all the terms of sequence, then it is called as series.

eg: $0 + 7 + 26 + \ldots$

eg: $2 + 3 + 5 + 7 + 11 + \ldots$

eg: $1 - 3 + 9 - 27 + \ldots$

eg: $\frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \frac{7}{17} + \ldots$

> Sequence → 2, 4, 6, 8, 10, ...
> Series → 2 + 4 + 6 + 8 + 10 + ...

* **Golden Point:**

T_n → denotes the n^{th} term of any sequence

S_n → denotes the summation of n terms of any series

$$\underbrace{\underbrace{T_1 + T_2 + T_3 + T_4 + T_5 + \cdots + T_{n-1}}_{S_{n-1}} + T_n}_{S_n}$$

$S_1 = T_1$

$S_2 = T_1 + T_2$

$S_3 = T_1 + T_2 + T_3$

⋮

NOTE: For any Series $\boxed{S_n - S_{n-1} = T_n}$

(Nth Term) − (Prev Term) = constant → (common difference)

$T_{n+1} - T_n = d \; , n \geq 1$

eg: 1, 3, 5, 7, 9, 11 .... A P

$T_2 - T_1 = 3 - 1 = 2$ → common difference (d) = 2

$T_3 - T_2 = 5 - 3 = 2$ → first Term $(T_1) = 1$

standard appearance of an A.P is

$\underset{T_1}{a}, \underset{T_2}{(a+d)}, \underset{T_3}{(a+2d)}, \underset{T_4}{(a+3d)} \ldots\ldots \underset{T_n}{a+(n-1)d}$

a → first Term of A.P

d → common difference

$T_n = a + (n-1)d$

$= a - d + nd$

let $a - d = A$

$\boxed{T_n = A + nd}$ → linear expression in n

* n^{th} term of a sequence is a linear polynomial in n ⟺ "T_n" sequence is an A.P (Non-constant)

b'coz: $T_n = an + b$

$T_{n-1} = a(n-1) + b$

$T_n - T_{n-1} = a = \text{constant}$ ⟹ sequence is A.P with common difference 'd'

eg: $T_n = 2n + 5$. Find the nature of sequence.

↓

since T_n is a linear polynomial in 'n' it is an A.P with common difference = 2

NOTE:

(i) If $T_n = an + b$, then the series so formed in A.P

(ii) (a) If $d > 0$ ⟹ increasing A.P

(b) If $d < 0$ ⟹ decreasing A.P

(c) If $d = 0$ ⟹ all the terms remain same and is constant A.P.

10, 8, 6, 4, 2, 0, −2, - - - - $d = -2$ (dec. AP)

2, 2, 2, 2, 2, 2 - - - - - - $d = 0$ (constant AP)

* Summation of n terms of an AP

$$S_n = a + (a+d) + (a+2d) + \cdots + a+(n-1)d$$

$$S_n = a+(n-1)d + a+(n-2)d + \cdots + a$$

$$2S_n = [2a+(n-1)d] \times n$$

$$\Rightarrow S_n = \frac{n}{2}[2a+(n-1)d]$$

$$\Rightarrow S_n = \frac{n}{2}[a + a+(n-1)d] = \frac{n}{2}[a+l]$$

$1+2+3+\cdots+100$

$100+99+98+\cdots+1$

$\frac{101 \times 100}{2} = S$

$S = 5050$

l = last term $= a+(n-1)d$

⇒ Remember That

(i) sum of first n natural number is $\frac{n(n+1)}{2}$

OR

$1+2+3+4+5+ \cdots + (n-1) + n = \frac{n(n+1)}{2}$

(ii) sum of first n odd natural number is n^2

OR

$1+3+5+7+ \cdots$ n-terms $= n^2$

(iii) sum of first n even natural number is $n(n+1)$

OR

$2+4+6+8+10+ \cdots$ n-terms $= n(n+1)$

⇒ K^{th} term from end of an A.P:

2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22 no. of terms = 11

M-1

Find: T_4 from last $= T_{11-4+1}$ from begining

$= T_8$ from begining

$= 2 + (8-1)2$

$= 2 + 7 \times 2 = 16$

K^{th} term from end of an A.P $= (n-K+1)^{th}$ term from begining

M-II To Find K^{th} term from ...

- Reverse the A.P i.e last term becomes first term
- New common difference becomes –ve of initial 'd'.
- Now determine K^{th} term from begining in this new A.P.

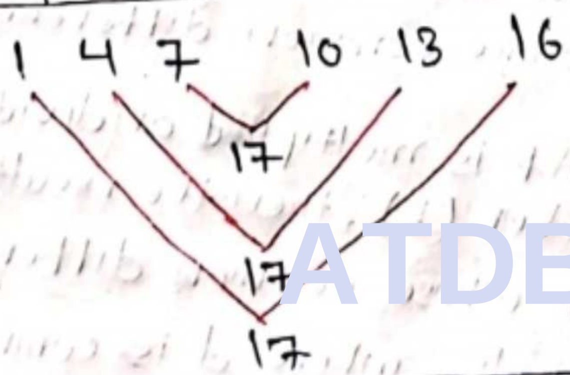
eg. Reverse the Previous eg. A.P

22, 20, 18, 16, 14, 12, 10, 8, 6, 4, 2 $(d=-2)$

Now T_4 from End $= T_4$ from begining in new. AP

$$= 22 + (4-1)(-2)$$
$$= 22 - 6 = \underline{16} \text{ Ans}$$

* An Important Fact about A.P



In A.P summation of K^{th} term from begining and K^{th} term from the last is always constant which is equal to summation of first term and last term.

$$\boxed{T_K + T_{n-K+1} = \text{constant} = a + l}$$

Proof T_K (From begining) $+ T_K'$ (From end) $=$ constant $= a + l$

$(l, l-d, l-2d, \ldots)$

$\underbrace{a+(K-1)d}_{T_K}$, $\underbrace{l+(K-1)(-d)}_{T_K'}$ $\Rightarrow \left\{ T_K + T_K' = a + l \right\}$

(i) Three number in A.P: $a-d,\ a,\ a+d$ (CD = d)

Five number in A.P: $a-2d,\ a-d,\ a,\ a+d,\ a+2d$ (CD = d)

Four number in AP: $a-3d,\ a-d,\ a+d,\ a+3d$ (CD = 2d)

$\boxed{\text{If terms are taken as above then remember first term is not 'a'}}$

(ii) If we pick the term of an A.P in a particular interval, then picked sequence is also an A.P with common difference interval times the original common difference.

$$\overbrace{1,\ 3,\ 5,\ 7}^{6},\ \overbrace{9,\ 11,\ 13}^{6}\ ------$$

$$\underbrace{1,\ 3}_{2},\ \underbrace{3,\ 5}_{2},\ \underbrace{5,\ 7}_{2},\ \underbrace{7,\ 9}_{2},\ \underbrace{9,\ 11}_{2},\ \underbrace{11,\ 13}_{2}$$

interval = Fourth term − First term = 3

d_F = Final common difference = 6

d_i = Initial common difference = 2

$d_F = (\text{interval}) \times d_i$

$6 = 3 \times 2$

(iii) If $a_1, a_2, a_3, ----$ and $b_1, b_2, b_3, ----$ are two A.Ps then $a_1 \pm b_1,\ a_2 \pm b_2,\ a_3 \pm b_3, ----$ are also in A.P but $a_1 \cdot b_1,\ a_2 \cdot b_2 ---$ and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, -----$ may OR may not be in A.P.

(iv) (a) ~~If~~ each term of an A.P is increased or decreased by the same number then the resulting sequence is also an A.P having the same common difference

(b) If each term of an A.P is multiplied or divided by the same non-zero number (k), then the resulting sequence is also an A.P whose common difference is 'kd' and '$\frac{d}{k}$' respectively, where d is common difference OR original A.P equal to half the sum of ... from it.

$$T_r = \frac{T_{r-k} + T_{r+k}}{2}, \quad k < r$$

$T_{7-2} = T_5$ $\qquad$ $T_{7+2} = T_9$

2, 4, 6, 8, 10, 12, 14, 16, 18, 20, ...

T_7

Reason:

$T_{r-k} = T_r - kd$

$T_{r+k} = T_r + kd$

$T_{r-k} + T_{r+k} = 2T_r$

$$\Rightarrow \left\{ T_r = \frac{T_{r-k} + T_{r+k}}{2} \right\}$$

$$\therefore T_7 = \frac{T_{(7-2)} + T_{(7+2)}}{2}$$

$$\Rightarrow 14 = \frac{10 + 18}{2} = 14$$

(vi) For any series $T_n = S_n - S_{n-1}$. In a series if S_n is a quadratic function of n of type $an^2 + bn$ or T_n is a linear function of n, then the series is an A.P

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right) = an + \frac{n(n-1)}{2}d$$

$$= n^2\left(\frac{d}{2}\right) + n\left(a - \frac{d}{2}\right)$$

$$\boxed{\text{if sum of n term of a sequence is of type } S_n = an^2 + bn \text{ then surely the sequence is an A.P}}$$

if $S_n = an^2 + bn \longrightarrow$ A.P

$T_n = S_n - S_{n-1} = an^2 + bn - \left(a(n-1)^2 + b(n-1)\right)$

$\Rightarrow \quad = \cancel{an^2} + \cancel{bn} - \cancel{an^2} + 2an - a - \cancel{bn} + b$

$T_n = 2an - a + b$

Now $T_{n-1} = 2a(n-1) - a + b$

$T_n - T_{n-1} = 2a$ = constant $\Rightarrow$ It is an A.P (Proved).

common difference $= 2a$

$T_1 = S_1 = a + b$

eg. $S_n = 5n^2 + 2n$ find seq^n.

$\Rightarrow d = 2a = 2 \times 5 = 10$

$\Rightarrow T_1 = S_1 = 7$

$\therefore$ seq^n 7, 17, 27, ...

* Arithmetic Mean:

→ if a, b, c are in A.P the b is called single Arithmetic mean b/w a & c

$$b - c = c - b$$

$$\boxed{2b = a + c}$$

a, b, c are in A.P $\Longleftrightarrow$ $2b = a + c$

if $a, A_1, A_2, A_3, \dots, A_n, b$ are in A.P then $A_1, A_2, A_3 \dots A_n$ are called n.A.Ms b/w a & b

$$\underset{T_1}{a}, \underbrace{\underset{T_2}{A_1}, \underset{T_3}{A_2}, \underset{T_4}{A_3}, A_4, \dots A_n}_{n\text{-terms}}, \underset{T_{n+2}}{b} \quad \Big\}\ \text{No: of terms} = n+2$$

$$T_{n+2} = b = a + (n+2-1)d = b$$

$$(n+1)d = b - a$$

$$d = \frac{b-a}{n+1}$$

let us find sum of n. AMs

$$T_2 = A_1 = a + d$$

$$T_3 = A_2 = a + 2d$$

$$T_4 = A_3 = a + 3d$$

$$\vdots$$

$$T_{n+1} = A_n = a + nd$$

$$A_1 + A_2 + A_3 + \dots + A_n = na + (1+2+3+\dots+n)d$$

$$= na + \frac{n(n+1)}{2} d \qquad \left\{ \because d = \frac{b-a}{n+1} \right\}$$

$$= na + \left(\frac{n\cancel{(n+1)}}{2}\right)\left(\frac{b-a}{\cancel{n+1}}\right)$$

$$= na + \frac{n(b-a)}{2}$$

$$= \frac{2na + nb - na}{2} = \frac{na + nb}{2} = n\left(\frac{a+b}{2}\right)$$

$$A_1 + A_2 + A_3 + \dots + A_n = n\left(\frac{a+b}{2}\right)$$

$\therefore$ sum of A.Ms b/w a & b = n. (single A.M b/w a & b)

$\frac{7n+1}{4n+27}$. Find the ratio of their ...

$$\frac{S_n}{S'_n} = \frac{7n+1}{4n+27} = \frac{\frac{n}{2}(2a+(n-1)d)}{\frac{n}{2}(2A+(n-1)D)}$$

$$\Rightarrow \frac{a+\left(\frac{n-1}{2}\right)d}{A+\left(\frac{n-1}{2}\right)D} = \frac{7n+1}{4n+27}$$

$T_{11} = a+10d$

$T'_{11} = A+10D$

put $\frac{n-1}{2} = 10$

$\Rightarrow n=21$

$$\Rightarrow n=21$$

$$\rightarrow \frac{a+10d}{A+10D} = \frac{(7\times21)+1}{(4\times21)+27}$$

$$\Rightarrow \frac{T_{11}}{T'_{11}} = \frac{148}{111} \Rightarrow \boxed{\frac{T_{11}}{T'_{11}} = 4/3}$$

Q Let A.P (a;d) denotes the set of all the terms of an infinite arithmetic progression with first term a and common difference d>0. If AP(1;3) ∩ AP(2;5) ∩ AP(3;7) = AP(a;d) find a&d.

AP(1;3) 1, 4, 7, 10, ------ $d_1=3$

AP(2;5) 2, 7, 12, 17 ------ $d_2=5$

common A.P $d_4 = Lcm(3,5) = 15$

7, 22, 37, ------

AP(3;7) 3, 10, 17, 24, ------ $d_3=7$

Now we find common A.P in,

7, 22, 37, ------ $d_4=15$

3, 10, 17, 24, ------ $d_3=7$

$d = Lcm(15,7) = \underline{105}$

$T_n = T_m$

$\Rightarrow 7+(n-1)15 = 3+(m-1)7$

$\Rightarrow 15n-18 = 7m-4$

$\Rightarrow n = \frac{7m+4}{15} \Rightarrow m=8,$

$n = \frac{60}{15} = 4$

∴ First common term

$\Rightarrow 7+(4-1)15 = 52$

∴ Now sequence of common terms

52, 157, 262, ------

$a=52$ 105

$AP(a;d) = AP(52;105)$

* No term of G.P can be zero

* $\left(\frac{\text{Agli term}}{\text{Picli term}}\right) = \text{constant}$ (common ratio) (r)

→ **Standard Appearance**

G.P: $a, ar, ar^2, ar^3, \ldots\ldots$

{ a → first term of G.P
r → common ratio of G.P }

$$\{T_n = ar^{n-1}\}$$ → nth term of G.P

* **Sum of n terms of G.P**

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

$$r \cdot S_n = 0 + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$$

$$(1-r)S_n = a - ar^n \Rightarrow \boxed{S_n = \frac{a(1-r^n)}{1-r},\ r \neq 1}$$

Agar word G.P is not used toh $r = 0$ is possible

if $r = 1$

$S_n = a + a + \cdots$ n term = na

Jab word G.P ka use kiyaa jayay toh dhyaan rakhe $r \neq 0$

$$\therefore S_n = \begin{cases} \frac{a(1-r^n)}{1-r}, & r \neq 1 \\ na, & r = 1 \end{cases}$$

2, 2, 2, 2, ... is an AP & GP Both

→ **Sum of infinite G.P**

$$S = a + ar + ar^2 + ar^3 + \cdots \infty \quad (r \neq 1)$$

$$\text{sum of n terms} = \frac{a(1-r^n)}{1-r}$$

$$\therefore S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

if $\boxed{-1 < r < 1}$, $r \neq 0$
⇓
$n \to \infty,\ r^n \to 0$

$$\Rightarrow \boxed{S_\infty = \frac{a}{1-r} \text{ if } -1 < r < 1,\ r \neq 0}$$

इस condition में ∞ G.P को Define value आएगी।

(i) Find k^{th} term from End in G.P $a, ar, ar^2, \ldots, ar^{n-1}$

* Reverse the G.P i.e last term becomes first term
* Now new common ~~difference~~ ratio $= \frac{1}{r}$

$ar^{n-1}, ar^{n-2}, ar^{n-3}, \ldots, a$

* Now find k^{th} term from Begining in above G.P to get desired ans.

We can also will T_k(from end) $= T_{(n-k+1)}$ (from begining)

(ii) Threee numbers in G.P. : $\frac{a}{r}, a, ar \rightarrow C.R = r \quad T_1 = \frac{a}{r}$

Five numbers in G.P. : $a/r^2, \frac{a}{r}, ar, ar^2 \rightarrow CR = r \quad T_1 = \frac{a}{r^2}$

Four numbers in G.P : $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3 \rightarrow C.R = r^2, T_1 = \frac{a}{r^3}$

Six numbers in G.P : $\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$
$\rightarrow C.R = r^2, T_1 = \frac{a}{r^5}$

r^2 can be −ve as well
bcoz idhar common Ratior r +ve, −ve dono हो सकता है
नहीं है r^2 है ये common ratio

(iii) If each terms of G.P be raised to the same power, then resulting series is also a G.P

eg: 2, 4, 8, 16, 32

$2^2, 4^2, 8^2, 16^2, 32^2$ — G.P

$2^{-1}, 4^{-1}, 8^{-1}, 16^{-1}, 32^{-1} \rightarrow$ G.P

(iv) If each term of a G.P be multiplied or divided by the same non-zero quantity, then the resulting sequence is also a G.P.

eg. 2, 4, 8, 16, ---- G.P ($r = 2$)

(divide by 2) 1, 2, 4, 8, ---- G.P ($r = 2$)

(multiply by 2) 4, 8, 16, 32, ---- G.P ($r = 2$)

common ratio r_1 and r_2 respectively then $a_1b_1, a_2b_2, a_3b_3, \ldots$ and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ will also form a G.P common ratio will be $r_1 \cdot r_2$ and $\frac{r_1}{r_2}$ respectively

(VI) In a positive G.P every term (except first) is equal to square root of product of its two terms which are equidistant from it.

i.e. $\boxed{T_r = \sqrt{T_{r-k} \cdot T_{r+k}} \quad , k < r}$

$(2^2) \quad (2^2)$

$2, 4, 8, 16, (32), 64, 128$

$\sqrt{8 \times 128} = 32$

Proof

$T_{r-k} = \frac{T_r}{R^k}$

$T_{r+k} = T_r \cdot R^k$

$T_{r-k} \cdot T_{r+k} = \frac{T_r}{R^k} \cdot T_r \cdot R^k$

$\Rightarrow \left\{ T_r = \sqrt{T_{r-k} \, T_{r+k}} \right\}$

(VII) If $a_1, a_2, a_3, \ldots a_n$ is a G.P of non-zero, non-negative terms then $\log a_1, \log a_2, \ldots, \log a_n$ is an A.P and vice-versa.

Proof

$a, ar, ar^2, ar^3 (ar^4), \ldots$

$\log()$ $\{ \log a, \log(ar), \log(ar^2), \log(ar^3), \log(ar^4), \ldots$

$\Rightarrow \log(a), \log(a) + \log(r), \log(a) + 2\log(r), \log a + 3\log r \ldots$ is an AP with common difference $\log(r)$.

(VIII) if $a, a+d, a+2d, \ldots$ are in A.P

$x^a, x^{a+d}, x^{a+2d}, \ldots$ are in G.P.

* Geometric Mean :-

If a, b, c are three positive number in G.P then b is called the single geometric mean b/w a & c

$\therefore \frac{b}{a} = \frac{c}{b}$

$\Rightarrow \boxed{b^2 = ac} \rightarrow a, b, c$ are in G.P

$\downarrow$

$b = \sqrt{ac}$

$a, \underbrace{g_1, g_2, g_3, \ldots, g_n}_{n\text{ G.Ms b/w } a \,\&\, b}, b$ are in G.P

$b = T_{n+2} = ar^{(n+2)-1} = ar^{n+1}$

$r^{n+1} = \frac{b}{a}$ —(i)

$g_1 = ar$
$g_2 = ar^2$
$g_3 = ar^3$
$\vdots$
$g_n = ar^n$

$g_1 \cdot g_2 \cdot g_3 \cdots g_n = ar \cdot ar^2 \cdot ar^3 \cdots ar^n$
$= a^n \cdot r^{1+2+3+\cdots+n}$
$= a^n \cdot r^{\left(\frac{n(n+1)}{2}\right)}$
$= a^n \cdot (r^{n+1})^{\frac{n}{2}}$
$= a^n \cdot \left(\frac{b}{a}\right)^{n/2} = a^n \cdot \frac{b^{n/2}}{a^{n/2}}$
$= a^{n-\frac{n}{2}} \cdot b^{n/2}$
$= a^{n/2} \cdot b^{n/2} = (\sqrt{ab})^n$
$= (\text{single G.M})^n$

∴ product of n.GMs b/w a & b = (single G.M b/w a&b)^n

eg: $2, \underbrace{4, 8, 16}_{3\text{ G.Ms b/w } 2 \,\&\, 32}, 32$

product of 3.G.Ms $= 4 \times 8 \times 16 = (\sqrt{2\times 32})^3 = 512$
$\Downarrow$ 512 (single G.M)

MATHEMATICAL GYAAN

* Cauchy Schwarz inequality : (CS inequality)

$\vec{v}_1 = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$
$\vec{v}_2 = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{v}_1 \cdot \vec{v}_2 = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\vec{v}_1||\vec{v}_2|\cos\theta$ — min = −1 $(-|\vec{v}_1||\vec{v}_2|)$; max = 1 $(+|\vec{v}_1||\vec{v}_2|)$

$\Rightarrow -|\vec{v}_1||\vec{v}_2| \leq \vec{v}_1 \cdot \vec{v}_2 \leq |\vec{v}_1| \cdot |\vec{v}_2|$

$\Rightarrow -|\vec{v}_1||\vec{v}_2| \leq a_1 b_1 + a_2 b_2 + a_3 b_3 \leq |\vec{v}_1||\vec{v}_2|$

$\Rightarrow |a_1 b_1 + a_2 b_2 + a_3 b_3| \leq |\vec{v}_1||\vec{v}_2|$

$\Rightarrow (\,)^2 \quad (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \leq |\vec{v}_1|^2 |\vec{v}_2|^2$

equality holds iff $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

$\Rightarrow \boxed{(a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}$

$\Rightarrow y = \cos x \cdot \sin x + \cos x \sqrt{\sin^2 x + 3}$

$\Rightarrow \vec{v_1} = \cos x\,\hat{i} + \sqrt{\sin^2 x + 3}\,\hat{j}$

$\vec{v_2} = \sin x\,\hat{i} + \cos x\,\hat{j}$

By using C.S inequality

$-|\vec{v_1}||\vec{v_2}| \leq \vec{v_1}\cdot\vec{v_2} \leq |\vec{v_1}||\vec{v_2}|$

$-\sqrt{\cos^2 x + \sin^2 x + 3} \cdot \sqrt{\cos^2 x + \sin^2 x} \leq \cos x \cdot \sin x + \cos x \cdot \sqrt{\sin^2 x + 3} \leq \sqrt{\cos^2 x + \sin^2 x + 3} \cdot \sqrt{\cos^2 x + \sin^2 x}$

$$-2 \leq y \leq 2$$

$$\Rightarrow y \in [-2, 2]$$

→ Generalized Form of C.S inequality

If $a_1, a_2, a_3 \ldots a_n$, $b_1, b_2, b_3, \ldots, b_n$ are real numbers then

$$\boxed{(a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2}$$

A Golden Concept:-

Q Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $a_1 = a_2 = 1$ and $a_{n+2} = 2a_{n+1} + a_n$ for all $n \geq 1$. Then the value of $47 \sum_{n=1}^{\infty} \frac{a_n}{8^n}$ is equal to

Target: $47 \sum_{n=1}^{\infty} \frac{a_n}{8^n}$, $a_{n+2} = 2a_{n+1} + a_n$, $n \geq 1$, $a_1 = 1, a_2 = 1$

$$47\underbrace{\left(\frac{a_1}{8} + \frac{a_2}{8^2} + \frac{a_3}{8^3} - \infty\right)}_{S}$$

$$\Rightarrow 8^2 \cdot \frac{a_{n+2}}{8^n \cdot 8^2} = 8 \cdot 2 \cdot \frac{a_{n+1}}{8^n \cdot 8} + \frac{a_n}{8^n}$$

$$\Rightarrow 64\left(\frac{a_{n+2}}{8^{n+2}}\right) = 16\,\frac{a_{n+1}}{8^{n+1}} + \frac{a_n}{8^n}$$

$$64 \sum_{n=1}^{\infty}\left(\frac{a_{n+2}}{8^{n+2}}\right) = \sum_{n=1}^{\infty} 16\left(\frac{a_{n+1}}{8^{n+1}}\right) + \sum_{n=1}^{\infty} \frac{a_n}{8^n}$$

$$\Rightarrow 64\left[\left(\frac{a_1}{8}+\frac{a_2}{8^2}+\frac{a_3}{8^3}+\frac{a_4}{8^4}+\cdots\infty\right)-\frac{a_1}{8}-\frac{a_2}{8^2}\right]=16\left[\left(\frac{a_1}{8}+\frac{a_2}{8^2}+\frac{a_3}{8^3}+\cdots\infty\right)-\frac{a_1}{8}\right]+S$$

$$\Rightarrow 64\left(S-\frac{1}{8}-\frac{1}{8^2}\right)=16\left(S-\frac{1}{8}\right)+S$$

$$\Rightarrow 64S-8-1=16S-2+S$$

$$\Rightarrow 47S=9-2=7$$

$$\Rightarrow 47S=7$$

Q $S=9+99+999+\cdots\cdots+99999\ldots9$ n times.

$$S=(10-1)+(10^2-1)+(10^3-1)+\cdots+(10^n-1)$$

$$=(10+10^2+10^3+\cdots+10^n)-n$$

$$=\frac{10(10^n-1)}{10-1}-n$$

$$=\frac{10}{9}(10^n-1)-n$$

* Arithmetic geometric progression (AGP):

standard appearance of an AGP is

$S=a,\ (a+d)r,\ (a+2d)r^2,\ (a+3d)r^3,\ \cdots$

Here each term is the product of corresponding terms in a arithmetic and geometric series.

Q if $|x|<1$, then compute S_∞: (AGP)

$1+2x+3x^2+4x^3+\cdots\infty$

$$\Rightarrow S=1+2x+3x^2+4x^3+\cdots\infty$$

$$xS=0+x+2x^2+3x^3+\cdots\infty$$

$$(1-x)S=1+x+x^2+x^3+\cdots\infty$$

$$(1-x)S=\frac{1}{1-x}$$

$$\Rightarrow \boxed{S=\frac{1}{(1-x)^2}}$$

A non-zero sequence is said to be in H.P if the reciprocals of ~~the~~ its terms are in A.P.

eg: If $a_1, a_2, a_3, ----$ are in H.P, then $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, ---$ are in A.P

A standard H.P is

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, ------, \frac{1}{a+(n-1)d}$$

$$\boxed{T_n = \frac{1}{a+(n-1)d}}$$

→ No term of H.P is zero

→ There is no formula for sum of H.P

Har H.P koa reciprocal A.P hogaa But reciprocal of every A.P may not be H.P b'coz some term of A.P may be zero

2) If a, b, c are in H.P $\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$\therefore \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \Rightarrow \boxed{b = \frac{2ac}{a+c}}$ → single H.M. b/w a & c

OR

$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b} \Rightarrow \frac{a-b}{ab} = \frac{b-c}{bc} \Rightarrow \boxed{\frac{a}{c} = \frac{a-b}{b-c}}$

⇒ Insertion of 'n' H.M's b/w a & b:

$a, H_1, H_2, H_3, ---, H_n, b \longrightarrow$ A.P

$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, ---, \frac{1}{H_n}, \frac{1}{b}$ are in A.P

n AM's b/w $\frac{1}{a}$ & $\frac{1}{b}$

we know sum of n-AMs = n (single A.M)

$\frac{1}{H_1} + \frac{1}{H_2} + --- + \frac{1}{H_n} = n\left(\frac{\frac{1}{a}+\frac{1}{b}}{2}\right) = n\left(\frac{a+b}{2ab}\right)$

Sum of reciprocals of n-HM's b/w a & b = n·(Reciprocal of single H.M. b/w a & b)

Sigma Notations ($\sum$) :

(1) $\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$

(2) $\sum_{r=1}^{n} K a_r = K\sum_{r=1}^{n} a_r$

(3) $\sum_{r=1}^{n} K = nK$; where K=constant

$$S = \sum_{r=1}^{n} f(r) = f(1)+f(2)+f(3)+ \cdots + f(n)$$

$$P = \prod_{r=1}^{n} f(r) = f(1)\cdot f(2)\cdot f(3) \cdots\cdots f(n)$$

* $\sum_{r=1}^{n} r$ = sum of first n natural numbers $= 1+2+3+\cdots+n = \frac{n(n+1)}{2}$

* $\sum_{r=1}^{n} r^2$ = sum of the squares of the first n natural numbers $= 1^2+2^2+3^2+\cdots+n^2 = \frac{n(2n+1)(n+1)}{2}$

* $\sum_{r=1}^{n} r^3$ = sum of the cubes of the first n natural numbers $= 1^3+2^3+3^3+\cdots+n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$

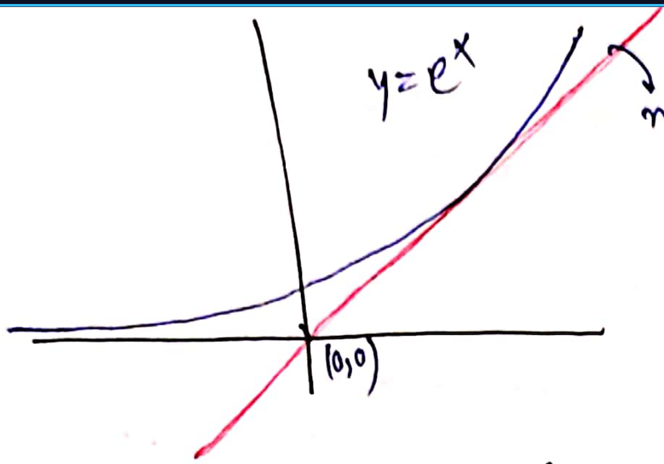
* $\sum_{r=1}^{n} (2r-1)$ = Sum of first n odd natural numbers $= 1, 3, 5, 7, 9, \cdots = n^2$

* $\sum_{r=1}^{n} 2r$ = Sum of first n even natural numbers $= 2, 4, 6, 8, \cdots = n(n+1)$

Agar koi bhi series ka hum T_r (r^{th} term) find karde toh uskaa sum likhna bahut aasaan ho jaata hai b'coz

$$S_n = T_1 + T_2 + \cdots + T_n$$

$$S_n = \sum_{r=1}^{n} T_r$$



{ slope of tangent drawn from origin to graph of $y = e^x$ is e }

Used when difference of consecutive terms are in A.P or G.P

* Method of Difference :-

Type 1: (Using method of Difference)

If $T_1, T_2, T_3, \ldots$ are the terms of a sequence then the terms $T_2 - T_1,\ T_3 - T_2,\ T_4 - T_3, \ldots$

Q $\underbrace{6+13}_{7}\underbrace{+22}_{9}\underbrace{+33}_{11} + \ldots\ n$ terms AP

r terms

$S = 6 + 13 + 22 + 33 \ldots + T_r$

$S = 6 + 13 + 22 + \ldots + T_{r-1} + T_r$

$S = \quad\ \ 6 + 13 + 22 \ldots + T_{r-1} + T_r$

$0 = 6 + (7 + 9 + 11 + \ldots \text{upto } (r-1) \text{ terms}) - T_r$

$\Rightarrow T_r = 6 + \frac{r-1}{2}\left(2\times 7 + (r-1-1)2\right)$

$= 6 + \frac{r-1}{2}(14 + 2r - 4)$

$= 6 + \frac{r-1}{2}(2r + 10) = 6 + (r-1)(r+5)$

$T_r = r^2 + 4r - 5 + 6 =$

$\{T_r = r^2 + 4r + 1\}$

Now. $S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n}(r^2 + 4r + 1)$

$= \sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$

$$\left\{ S_n = \frac{n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} + n \right\}$$

$2 \quad 6 \quad 18 \quad 54 \; - - -$

$$S = 5 + 7 + 13 + 31 + 85 + \cdots + T_r$$

$$S = \quad 5 + 7 + 13 + 31 + \cdots + T_{r-1} + T_r$$

$$0 = 5 + (2 + 6 + 18 + 54 + \ldots \text{upto } (r-1) \text{ terms}) - T_r$$

$$\Rightarrow T_r = 5 + (2 + 6 + 18 + \ldots \text{upto } (r-1) \text{ terms})$$

$$= 5 + \frac{2\cdot(3^{r-1}-1)}{3-1} = 5 + 3^{r-1} - 1 = 3^{r-1} + 4$$

$$\Rightarrow T_r = 3^{r-1} + 4$$

$$S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (3^{r-1} + 4)$$

$$= \sum_{r=1}^{n} 3^{r-1} + 4n$$

$$= (3^0 + 3^1 + 3^2 + \cdots \; n \text{ terms}) + 4n$$

$$= \frac{1\cdot(3^n - 1)}{2} + 4n$$

$$\left\{ S_n = \frac{3^n - 1}{2} + 4n \right\}$$

* Another OR Alternative Method

$$f(n) = an^2 + bn + c$$

$$f(n-1) = a(n-1)^2 + b(n-1) + c$$

$$f(n) - f(n-1) = \cancel{an^2} + \cancel{bn} + \cancel{c} - (\cancel{an^2} - 2an + a + \cancel{bn} - b + \cancel{c})$$

$$g(n) = f(n) - f(n-1) = 2an + b - a \longrightarrow \text{1st order diff}$$

$$g(n-1) = f(n-1) - f(n-2) = 2a(n-1) + b - a$$

$$g(n) - g(n-1) = 2a = \text{constant} \longrightarrow \text{2nd order diff.}$$

"2nd degree polynomial ... hota hain"

"3rd degree polynomial kaa 3rd order difference constant hota hain"

"nth degree polynomial kaa nth order difference constant hota hain"

NOTE:

If k^{th} order difference between consecutive terms is constant then r^{th} term is polynomial of degree k in r.

eg. (1) 1, 3, 5, 7, ...

1st order difference $d_1 = 2$ (constant)

$\Rightarrow T_r = ar + b$

$T_1 = a + b = 1$

$T_2 = 2a + b = 3$

$a = 2,\ b = -1$

$a = 2,\ b = 1$

$\left\{ \therefore T_r = 2r - 1 \right\}$

(2) 3, 7, 14, 24, ... n terms

1st order difference d_1: 4, 7, 10 ...

2nd order difference d_2: 3 (constant)

$\Rightarrow T_r = ar^2 + br + c$

$T_1 = a + b + c = 3$

$T_2 = 4a + 2b + c = 7$

$T_3 = 9a + 3b + c = 14$

$T_2 - T_1 = 3a + b = 4$

$T_3 - T_2 = 5a + b = 7$

$2a = 3$

$\Rightarrow a = 3/2$

$b = 4 - \frac{9}{2}$

$\Rightarrow -1/2$

$c = 3 - 3/2 + 1/2$

$= 2$

$a = 3/2,\ b = -1/2,\ c = 2$

$\therefore T_r = \frac{3}{2} r^2 - \frac{1}{2} r + 2$

if we find S_n

$\therefore$ we find $\sum_{r=1}^{n} T_r$

$\therefore \sum_{r=1}^{n} \left(\frac{1}{2} r^2 - \frac{1}{2} r + 2 \right)$

we get S_n

$$f(n-1) = ax^{n-1} + b(n-1) + c$$

$$g(n) = f(n) - f(n-1) = a(x^n - x^{n-1}) + b \rightarrow \text{1st order difference}$$

$$g(n) = ax^{n-1}(x-1) + b$$

$$g(n-1) = ax^{n-2}(x-1) + b$$

$$g(n) - g(n-1) = a(x^{n-1} - x^{n-2})(x-1) = a(x-1)^2 x^{n-2} \rightarrow \text{2nd order difference}$$

$\hookrightarrow$ G.P $\rightarrow$ common ratio $= x$

* If k^{th} order difference are in G.P then r^{th} term of the sequence is given by $T_r = a(CR)^r +$ a polynomial of degree $(k-1)$ in 'n'

eg (1) 1, 3, 7, 15 ....

1st order difference $d_1 = 2, 4, 8, \ldots$

$\Rightarrow T_r = a(2)^r + b$

$T_1 = 2a + b = 1$

$T_2 = 4a + b = 3$

$2a = 2 \Rightarrow \boxed{a=1}$ $\boxed{b=-1}$

since $a = 1$, $b = -1$

$$\left\{ \therefore T_r = 2^r - 1 \right\}$$

Q Find the sum of n-terms of the series

$$3 + 8 + 22 + 72 + 266 + 1036 \ldots\ldots$$

$5 \quad 14 \quad 50 \quad 194 \quad 770 \ldots$: 1st order difference

$9 \quad 36 \quad 144 \quad 576 \ldots\ldots$ 2nd order difference is in G.P.

$\therefore T_r = a(4)^r + br + c$

$T_1 = 4a + b + c = 3$
$T_2 = 16a + 2b + c = 8$
$T_3 = 64a + 3b + c = 22$

$12a + b = 5$
$48a + b = 14$

$36a = 9 \Rightarrow (a = 1/4)$
$(b = 2)$
$(c = 0)$

$\therefore a = 1/4$, $b = 2$, $c = 0$

$$\therefore T_r = \frac{1}{4}4^r + 2r = 4^{r-1} + 2r$$

$$S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} 4^{r-1} + 2r$$

$$= \sum_{r=1}^{n} 4^{r-1} + 2\sum_{r=1}^{n} r$$

$$S_n = (4^0 + 4^1 + 4^2 + \cdots + 4^{n-1}) + \frac{2n(n+1)}{2}$$

$$\left\{ S_n = \frac{4^n - 1}{3} + n(n+1) \right\}$$

* if k^{th} order diff is const.

$\Rightarrow$ T_r = polynomial of degree k

* if k^{th} order diff is in G.P

$\Rightarrow T_r = a\cdot(\text{common Ratio})^r +$ a polynomial of degree $(k-1)$

Q find S_n for —

$$1+\left(1+\frac{1}{2}+\frac{1}{2^2}\right)+\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}\right)+\cdots\cdots$$

$$S_n = 1+\frac{1\cdot\left(1-\frac{1}{2^3}\right)}{1-1/2}+\frac{1\cdot\left(1-\frac{1}{2^5}\right)}{1-1/2}+\cdots$$

$$= 1+\frac{1-1/2^3}{1/2}+\frac{1-1/2^5}{1/2}+\frac{1-1/2^7}{1/2}+\cdots$$

$$= \frac{1-1/2}{1/2}+\frac{1-1/2^3}{1/2}+\frac{1-1/2^5}{1/2}+\frac{1-1/2^7}{1/2}+\cdots \text{up to } n \text{ terms}$$

$$= \frac{1-1/2+1-1/2^3+1-1/2^5+1-1/2^7+\cdots}{1/2}$$

$$= \frac{n-\left(1/2+1/2^3+1/2^5+1/2^7\cdots \text{up to } n \text{ terms}\right)}{1/2}$$

$$\therefore \left\{ S_n = \frac{n-\frac{1/2\left(1-(1/4)^n\right)}{1-1/4}}{1/2} \right\}$$

Type 2: (Splitting the n^{th} term as a difference)

There is a series in which each term in composed of the reciprocal of the product of r factors in A.P, the first factor of the several terms beings in the same A.P

Q $\frac{1}{1\cdot2\cdot3\cdot5}+\frac{1}{2\cdot3\cdot4\cdot5}+\frac{1}{3\cdot4\cdot5\cdot6}+$ .... up to n terms

MANTRA!! continued product barbad kar Honay dungaaa

$$T_r=\frac{1}{r(r+1)(r+2)(r+3)}$$

continued product.

$$=\frac{(r+3)-r}{3\cdot r(r+1)(r+2)(r+3)}$$

$$T_r=\frac{1}{3}\left[\frac{1}{r(r+1)(r+2)}-\frac{1}{(r+1)(r+2)(r+3)}\right]$$

$$T_1=\frac{1}{3}\left[\frac{1}{1\cdot2\cdot3}-\cancel{\frac{1}{2\cdot3\cdot4}}\right]$$

$$T_2=\frac{1}{3}\left[\cancel{\frac{1}{2\cdot3\cdot4}}-\cancel{\frac{1}{3\cdot4\cdot5}}\right]$$

$$T_3=\frac{1}{3}\left[\cancel{\frac{1}{3\cdot4\cdot5}}-\cancel{\frac{1}{4\cdot5\cdot6}}\right]$$

$$\vdots$$

$$T_n=\frac{1}{3}\left[\cancel{\frac{1}{n(n+1)(n+2)}}-\frac{1}{(n+1)(n+2)(n+3)}\right]$$

$$\Rightarrow S_n=\frac{1}{3}\left[\frac{1}{1\cdot2\cdot3}-\frac{1}{(n+1)(n+2)(n+3)}\right]$$

If Question is Find Sum of ∞ terms :-

$$S_\infty=\lim_{n\to\infty}S_n=\frac{1}{3}\left[\frac{1}{6}\right]$$

$$\left\{\therefore S_\infty=\frac{1}{18}\right\}$$

denominator we multiply ... that the In denominator we get a complete continued product

Q $\frac{3}{1\cdot2\cdot4}+\frac{4}{2\cdot3\cdot5}+\frac{5}{3\cdot4\cdot6}+\cdots\cdots$

$T_r=\frac{r+2}{r(r+1)(r+3)}$ ← Not a continued product $(r+2)$ is missing.

$T_r=\frac{(r+2)(r+2)}{r(r+1)(r+2)(r+3)}=\frac{(r+2)^2}{r(r+1)(r+2)(r+3)}$

$=\frac{r^2+4r+4}{r(r+1)(r+2)(r+3)}=\frac{r^2+3r+r+3+1}{r(r+1)(r+2)(r+3)}$

$=\frac{r(r+3)+(r+3)+1}{r(r+1)(r+2)(r+3)}$

$\therefore T_r=\frac{1}{(r+1)(r+2)}+\frac{1}{r(r+1)(r+2)}+\frac{1}{r(r+1)(r+2)(r+3)}$

$=\frac{(r+2)-(r+1)}{(r+1)(r+2)}+\frac{r+2-r}{2r(r+1)(r+2)}+\frac{(r+3)-r}{3r(r+1)(r+2)(r+3)}$

$T_r=\left(\frac{1}{r+1}-\frac{1}{r+2}\right)+\frac{1}{2}\left(\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}\right)$
$+\frac{1}{3}\left[\frac{1}{r(r+1)(r+2)}-\frac{1}{(r+1)(r+2)(r+3)}\right]$

$\therefore T_1=\left(\frac{1}{2}-\cancel{\frac{1}{3}}\right)+\frac{1}{2}\left(\frac{1}{1\cdot2}-\cancel{\frac{1}{2\cdot3}}\right)+\frac{1}{3}\left(\frac{1}{1\cdot2\cdot3}-\cancel{\frac{1}{2\cdot3\cdot4}}\right)$

$T_2=\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\frac{1}{2}\left(\cancel{\frac{1}{2\cdot3}}-\cancel{\frac{1}{3\cdot4}}\right)+\frac{1}{3}\left(\cancel{\frac{1}{2\cdot3\cdot4}}-\cancel{\frac{1}{3\cdot4\cdot5}}\right)$

$T_3=\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{5}}\right)+\frac{1}{2}\left(\cancel{\frac{1}{3\cdot4}}-\frac{1}{4\cdot5}\right)+\frac{1}{3}\left(\cancel{\frac{1}{3\cdot4\cdot5}}-\frac{1}{4\cdot5\cdot6}\right)$

$\vdots$

$T_n=\left(\cancel{\frac{1}{n+1}}-\frac{1}{n+2}\right)+\frac{1}{2}\left(\cancel{\frac{1}{n(n+1)}}-\frac{1}{(n+1)(n+2)}\right)+\frac{1}{3}\left(\cancel{\frac{1}{n(n+1)(n+2)}}-\frac{1}{(n+1)(n+2)(n+3)}\right)$

$S_n=\sum_{i=1}^{n}T_i=\left(\frac{1}{2}-\frac{1}{n+2}\right)+\frac{1}{2}\left(\frac{1}{1\cdot2}-\frac{1}{(n+1)(n+2)}\right)$
$+\frac{1}{3}\left(\frac{1}{1\cdot2\cdot3}-\frac{1}{(n+1)(n+2)(n+3)}\right)$

if Question is Find S_∞ then

$S_\infty=\frac{1}{2}+\frac{1}{2}\left(\frac{1}{1\cdot2}\right)+\frac{1}{3}\left(\frac{1}{6}\right)$

$$T_r = \frac{r}{1\cdot3\cdot5----(2r+1)} = \frac{(2r+1)-1}{2\cdot(1\cdot3\cdot5----(2r+1))}$$

$$= \frac{(2r+1)-1}{2(1\cdot3\cdot5---(2r-1)(2r+1))}$$

$$T_r = \frac{1}{2}\left\{\frac{1}{1\cdot3\cdot5\cdots(2r-1)} - \frac{1}{1\cdot3\cdot5---(2r+1)}\right\}$$

$$T_1 = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{1\cdot3}\right)$$

$$T_2 = \frac{1}{2}\left(\frac{1}{1\cdot3} - \frac{1}{1\cdot3\cdot5}\right)$$

$$T_3 = \frac{1}{2}\left(\frac{1}{1\cdot3\cdot5} - \frac{1}{1\cdot3\cdot5\cdot7}\right)$$

$$\vdots$$

$$T_n = \frac{1}{2}\left(\frac{1}{1\cdot3\cdot5--(2n-1)} - \frac{1}{1\cdot3\cdot5--(2n+1)}\right)$$

$$\therefore S_n = \sum_{i=1}^{n} T_i = \frac{1}{2}\left(1 - \frac{1}{1\cdot3\cdot5--(2n+1)}\right)$$

***Type 3 :** Here is a series in which each term in composed of r factors in A.P, the first factor of the several terms beings in the same A.P.

Q $1\cdot2\cdot3\cdot4 + 2\cdot3\cdot4\cdot5 + 3\cdot4\cdot5\cdot6 + ----$ up to n terms.

$(r-1)$ [Previous term]

$$T_r = \frac{1}{5} r(r+1)(r+2)(r+3)\left[(r+4)-(r-1)\right]$$

$(r+4)$ [Next term]

$$T_r = \frac{1}{5}\left[r(r+1)(r+2)(r+3)(r+4) - (r-1)r(r+1)(r+2)(r+3)\right]$$

$$T_1 = \frac{1}{5}\left[1\cdot2\cdot3\cdot4\cdot5 - 0\right]$$

$$T_2 = \frac{1}{5}\left[2\cdot3\cdot4\cdot5\cdot6 - 1\cdot2\cdot3\cdot4\cdot5\right]$$

$$T_3 = \frac{1}{5}\left[3\cdot4\cdot5\cdot6\cdot7 - 2\cdot3\cdot4\cdot5\cdot6\right]$$

$$\vdots$$

$$T_n = \frac{1}{5}\left[n(n+1)(n+2)(n+3)(n+4) - (n-1)n(n+1)(n+2)(n+3)\right]$$

$$S_n = \frac{1}{5}\left(n(n+1)(n+2)(n+3)(n+4)\right)$$

$T_r = r(r+1)(r+4)$ → Not a continued Product

$= r(r+1)((r+2)+(2)) = r(r+1)(r+2) + 2\cdot r(r+1)$

$= \frac{1}{4} r(r+1)(r+2)\left[(r+3)-(r-1)\right] + \frac{2r(r+1)}{3}\left[(r+2)-(r-1)\right]$

$$T_r = \frac{1}{4}\left[r(r+1)(r+2)(r+3) - (r-1)r(r+1)(r+2)\right] + \frac{2}{3}\left[r(r+1)(r+2) - (r-1)r(r+1)\right]$$

$$T_1 = \frac{1}{4}\left(1\cdot2\cdot3\cdot4 - 0\right) + \frac{2}{3}\left(1\cdot2\cdot3 - 0\right)$$

$$T_2 = \frac{1}{4}\left(2\cdot3\cdot4\cdot5 - 1\cdot2\cdot3\cdot4\right) + \frac{2}{3}\left(2\cdot3\cdot4 - 1\cdot2\cdot3\right)$$

$$T_3 = \frac{1}{4}\left(3\cdot4\cdot5\cdot6 - 2\cdot3\cdot4\cdot5\right) + \frac{2}{3}\left(3\cdot4\cdot5 - 2\cdot3\cdot4\right)$$

$$T_n = \frac{1}{4}\left(n(n+1)(n+2)(n+3) - (n-1)n(n+1)(n+2)\right) + \frac{2}{3}\left(n(n+1)(n+2) - (n-1)n(n+2)\right)$$

$$S_n = \frac{1}{4}\left(n(n+1)(n+2)(n+3)\right) + \frac{2}{3}\left(n(n+1)(n+2)\right)$$

* A.M, G.M and H.M. of two positive numbers

let a & b be two positive no!s

a, A, b are in A.P. ⇒ single A.M b/w a&b = $A = \frac{a+b}{2}$

a, G, b are in G.P ⇒ single G.M b/w a&b = $G = \sqrt{ab}$

a, H, b are in G.P ⇒ single ⇒ single H.M b/w a&b = $H = \frac{2ab}{a+b}$

$$G^2 = ab = \left(\frac{a+b}{2}\right)\cdot\left(\frac{2ab}{a+b}\right) = AH$$

$\boxed{\therefore\ G^2 = A\cdot H}$ (This ~~not hold~~ does not hold for n number only for two +ve numbers)

$\left(\text{single G.M b/w two numbers}\right)^2 = \left(\text{A.M b/w the two No!s}\right)\cdot\left(\text{H.M b/w the two no!s}\right)$

the AM, GM and HM b/w the numbers ... the GM exceeds HM by 4, then (where $A>1, G>1, H>1$)

(A) $A+G=30H$ (B) $G+H=A+11$ (C) $4(G+H)=A$ (D) $A+G=3(H-1)$

$$a+b=50 \Rightarrow A=\frac{a+b}{2}=25$$

ATQ

G.M exceed H.M by 4

$$G=H+4$$

$$\Rightarrow H=G-4$$

Now $G^2=A\cdot H \Rightarrow G^2=25(G-4) \Rightarrow G^2-25G+100=0$

$$\Rightarrow G^2-20G-5G+100=0$$

$$\Rightarrow G(G-20)-5(G-20)=0$$

$$\Rightarrow G=5,\ 20$$

$$H=\frac{25}{25},\ \frac{(20)^2}{25} \quad \left\{\because H=\frac{G^2}{A}\right\}$$

$$=1,\ 16$$

Becaus in question it is given $H>1$

$\therefore$ $A=25$, $G=20$, $H=16$

{since option (B) & option (D) is correct}

* A.M, G.M & H.M of 'n' numbers:-

*** A.M**

$$A_n=\frac{\sum_{k=1}^{n} a_k}{n}=\frac{a_1+a_2+a_3+\cdots+a_n}{n}$$

*** G.M**

$$G_n=\left(\prod_{k=1}^{n} a_k\right)^{1/n}=(a_1\cdot a_2\cdot a_3\cdots\cdots a_n)^{1/n}$$

*** H.M**

$$H_n=\frac{n}{\sum_{k=1}^{n}\frac{1}{a_k}}=\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}}$$

$$(\sqrt{a}-\sqrt{b})^2 \geq 0$$

$$\Rightarrow a+b-2\sqrt{ab} \geq 0$$

$$\Rightarrow \frac{a+b}{2} \geq \sqrt{ab}$$

$$\Rightarrow AM \geq GM$$

$$\boxed{A \geq G} \Rightarrow \frac{A}{G} \geq 1$$

Now $G.M^2 = (A.M)\cdot(G.M)$

i.e $G^2 = A.H$

$$\Rightarrow \frac{G}{H} = \frac{A}{G} \geq 1 \quad \left\{\because \frac{A}{G} \geq 1\right\}$$

$$\Rightarrow \frac{G}{H} \geq 1$$

$$\therefore \boxed{G \geq H}$$

* Root Mean Square

$$\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}$$

↓

root mean square

OR

(R.M.S)

If a, b are two +ve NO:S are equale then

AM = GM = H.M

* Generalise

If $a_1, a_2, a_3, \ldots, a_n$ are n +ve reals then

$$\boxed{R.MS \geq A.M \geq G.M \geq H.M}$$

$$\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}} \geq \frac{a_1+a_2+\cdots+a_n}{2} \geq (a_1\cdot a_2\cdots a_n)^{1/n} \geq \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}$$

Q a, b, c > 0 then Prove $a^2(b+c)+b^2(c+a)+c^2(a+b) \geq 6abc$

$S = a^2(b+c)+b^2(c+a)+c^2(a+b)$

$= a^2b+a^2c+b^2c+b^2a+c^2a+c^2b$

APPLYING AM, GM ineq. (AM ≥ GM)

$$\Rightarrow \frac{a^2b+a^2c+b^2c+b^2a+c^2a+c^2b}{6} \geq (a^2b\cdot a^2c\cdot b^2c\cdot b^2a\cdot c^2a\, c^2b)^{1/6}$$

$$\Rightarrow \frac{S}{6} \geq (a^6b^6c^6)^{1/6} \Rightarrow \boxed{S \geq 6abc}$$

QED

$$\Rightarrow (a^3+a^2+a+1)^2 \geq 16a^3$$

Proof By AM≥GM

$$\frac{a^3+a^2+a+1}{4} \geq (a^3\cdot a^2\cdot a\cdot 1)^{1/4}$$

$$\Rightarrow a^3+a^2+a+1 \geq 4a^{6/4}$$

$$\Rightarrow a^3+a^2+a+1 \geq 4a^{3/2}$$

$$(\;)^2 \quad (a^3+a^2+a+1)^2 \geq 16a^3 \quad (QED)$$

Q If $x>0, y>0, z>0$ then prove that $(x+y)(y+z)(z+x) \geq 8xyz$

$$\frac{x+y}{2} \geq \sqrt{xy}$$

$$\frac{y+z}{2} \geq \sqrt{yz}$$

$$\frac{z+x}{2} \geq \sqrt{zx}$$

$$\frac{(x+y)(y+z)(z+x)}{8} \geq \sqrt{x^2\cdot y^2\cdot z^2}$$

$$\Rightarrow (x+y)(y+z)(z+x) \geq 8xyz \quad (QED)$$

Q Show that if a, b, c, d be four positive unequal quantities and $s = a+b+c+d$, then

$$(s-a)(s-b)(s-c)(s-d) > 81\,abcd$$

$\Rightarrow s-a = b+c+d$, $s-b = a+c+d$, $s-c = a+b+d$, $s-d = a+b+c$

$\therefore$ By AM > GM

$$\frac{b+c+d}{3} > \sqrt[3]{bcd}, \quad \frac{a+c+d}{3} > \sqrt[3]{acd}, \quad \frac{a+b+d}{3} > \sqrt[3]{abd}, \quad \frac{a+b+c}{3} > \sqrt[3]{abc}$$

Multiply all 4 inequality

$$\Rightarrow \frac{b+c+d}{3}\cdot\frac{a+c+d}{3}\cdot\frac{a+b+d}{3}\cdot\frac{a+b+c}{3} > \sqrt[3]{a^3\cdot b^3+c^3\cdot d^3}$$

$$\Rightarrow (b+c+d)(a+c+d)(a+b+d)(a+b+c) > 81\,abcd$$

$$\Rightarrow \{(s-a)(s-b)(s-c)(s-d) > 81\,abcd\}$$

$$\rightarrow e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \text{ ---- } \infty, \quad x \in R$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \text{ ---- } \infty, \quad x \in R$$

$$e^x + e^{-x} = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \infty\right)$$

$$e^x - e^{-x} = 2\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \text{---} \infty\right)$$

$\boxed{\text{If } x = 1}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\therefore \quad e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \text{---} \infty$$

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \text{---} \infty$$

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \text{----} \infty$$

$$e + e^{-1} = 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \text{--} \infty\right)$$

$$e - e^{-1} = 2\left(1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \text{--} \infty\right)$$

$$\rightarrow \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \text{ ---- } \infty, \quad -1 < x \le 1$$

$$\ln(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \text{---} \infty\right)$$

$$\ln(1-x^2) = -2\left(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \text{---} \infty\right)$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \text{----} \infty\right)$$

$$\ln(2) = \ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{---} \infty$$

(a) $\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \frac{8}{9!} + \cdots \infty$

$$T_r = \frac{(2r+1)-1}{(2r+1)!} = \frac{2r+1}{(2r+1)!} - \frac{1}{(2r+1)!}$$

$$= \frac{(2r+1)}{(2r+1)(2r)!} - \frac{1}{(2r+1)!}$$

$(2r)! \neq 2(r!)$

$$T_r = \frac{1}{(2r)!} - \frac{1}{(2r+1)!}$$

$$S = \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{4!} - \frac{1}{5!}\right) + \left(\frac{1}{6!} - \frac{1}{7!}\right) + \cdots \infty$$

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots$$

$\{S = e^{-1}\}$ Answer

(b) $\frac{5}{1!} + \frac{9}{2!} + \frac{13}{3!} + \frac{17}{4!} + \cdots \infty = 5e - 1$ (Proof)

$5, 9, 13, 17 \cdots$
$T_r = 5 + (r-1)\cdot 4 = 4r+1$

$$T_r = \frac{4r+1}{r!} = \frac{4r}{r!} + \frac{1}{r!}$$

$$= \frac{4r}{r(r-1)!} + \frac{1}{r!}$$

$$T_r = \frac{4}{(r-1)!} + \frac{1}{r!}$$

$$S = 4\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \infty\right) + \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \infty\right)$$

$$= 4\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \infty\right) + \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \infty - 1\right)$$

$$= 4e + e - 1$$

$\{S = 5e - 1\}$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \infty$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty$$

Find sum of $\sum_{n=1}^{\infty}\left(\frac{2S_n}{n!}-\frac{1}{(n-2)!}\right)$ {JEE MAIN 2021}

$$S_n = 1\cdot(n-1)+2\cdot(n-2)+3\cdot(n-3)+\cdots+(n-1)\cdot 1$$

$$T_r = r(n-r) = nr - r^2$$

$$S_n = \sum_{r=1}^{n-1} T_r = n\sum_{r=1}^{n-1} r - \sum_{r=1}^{n-1} r^2$$

$$= n\cdot\frac{(n-1)n}{2} - \frac{(n-1)n(2n-1)}{6}$$

$\left\{ \because 1+2+3+\cdots+n = \frac{n(n+1)}{2},\quad 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6} \right\}$

$$= \frac{n(n-1)}{2}\left[n - \frac{(2n-1)}{3}\right]$$

$$= \frac{n(n-1)(n+1)}{6}$$

ACQ

$$S = \sum_{n=4}^{\infty}\left(\frac{2S_n}{n!} - \frac{1}{(n-2)!}\right)$$

$$= \sum_{n=4}^{\infty}\left\{\frac{2\left(\frac{n(n-1)(n+1)}{6}\right)}{n!} - \frac{1}{(n-2)!}\right\}$$

$$= \sum_{n=4}^{\infty}\left\{\left(\frac{\cancel{n}\cancel{(n-1)}(n+1)}{3\cdot \cancel{n}\cancel{(n-1)}(n-2)!}\right) - \frac{1}{(n-2)!}\right\}$$

$$= \sum_{n=4}^{\infty}\left(\frac{n+1}{3(n-2)!} - \frac{1}{(n-2)!}\right) = \sum_{n=4}^{\infty}\left(\frac{(n-2)+3}{3(n-2)!} - \frac{1}{(n-2)!}\right)$$

$$= \sum_{n=4}^{\infty}\left(\frac{\cancel{n-2}}{3\cancel{(n-2)}(n-3)!} + \cancel{\frac{1}{(n-2)!}} - \cancel{\frac{1}{(n-2)!}}\right)$$

$$S = \frac{1}{3}\sum_{n=4}^{\infty}\frac{1}{(n-3)!} = \frac{1}{3}\left[\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots\infty\right]$$

$$S = \frac{1}{3}\left[1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots\infty - 1\right]$$

$$\left\{S = \frac{1}{3}(e-1) = \frac{e-1}{3}\right\}$$

$$S=\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}-\frac{1}{4\cdot 5}+---\infty$$

$$T_r=\frac{(-1)^{r+1}}{r(r+1)}=(-1)^{r+1}\left(\frac{r+1-r}{r(r+1)}\right)$$

$$=(-1)^{r+1}\left(\frac{1}{r}-\frac{1}{r+1}\right)$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}+--\infty$$

$$T_r=\left(\frac{(-1)^{r+1}}{r}-\frac{(-1)^{r+1}}{r+1}\right)$$

$$S=\left[\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}----\infty\right]-\left[\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+--\infty\right]$$

$$=\ln(1+1)+\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}---\infty-1\right)$$

$$=\ln(2)+(\ln(2)-1)$$

$$=2\ln(2)-1$$

Q if $x\in(0,1)$ then $\frac{3}{2}x^2+\frac{5}{3}x^3+\frac{7}{4}x^4----$ is equal to

$$S=\frac{3}{2}x^2+\frac{5}{3}x^3+\frac{7}{4}x^4---\infty$$

$$T_r=\frac{(2r+1)x^{r+1}}{(r+1)}=\left[\frac{(2r+2)-1}{r+1}\right]\cdot x^{r+1}$$

$$=\left(2-\frac{1}{r+1}\right)x^{r+1}=\left(2\cdot x^{r+1}-\frac{x^{r+1}}{r+1}\right)$$

$$S=2(x^2+x^3+x^4+--\infty)-\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+--\infty\right)-x$$

$$=\frac{2x^2}{1-x}+\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+--\infty\right)+x$$

$$=\frac{2x^2}{1-x}+\ln(1-x)+x$$

$$=\frac{2x^2+x-x^2}{1-x}+\ln(1-x)$$

$$S=\frac{x^2+x}{1-x}+\ln(1-x)=\frac{x(1+x)}{1-x}+\ln(1-x)$$ Ans

$$\ln(1-x)=-\left(x+\frac{x^2}{2}+\frac{x^2}{3}+\frac{x^4}{4}+--\infty\right)$$