

CHAPTER

8



Complex Number-I

IO TA

So, $i = \sqrt{-1}, i^2 = -1, i^3 = -i, i^4 = 1$
 $i^{4n+1} = i, i^{4n+2} = -1, i^{4n+3} = -i, i^{4n+4} = 1$

In other words, $i^n = \begin{cases} (-1)^{n/2}, & \text{if } n \text{ is an even integer} \\ (-1)^{\frac{n-1}{2}} \cdot i, & \text{if } n \text{ is an odd integer} \end{cases}$

The Complex Number System

$z = a + ib$, then $a - ib$ is called conjugate of z and is denoted by \bar{z}

Equality in Complex Number

$z_1 = z_2 \Rightarrow \text{Re}(z_1) = \text{Re}(z_2) \text{ and } \text{Im}(z_1) = \text{Im}(z_2)$

Conjugate Complex

If $z = a + ib$ then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by \bar{z} i.e. $\bar{z} = a - ib$.

Note:

- (i) $z + \bar{z} = 2 \text{Re}(z)$
- (ii) $z - \bar{z} = 2i \text{Im}(z)$
- (iii) $z\bar{z} = a^2 + b^2$ which is real
- (iv) If z is purely real then $z - \bar{z} = 0$
- (v) If z is purely imaginary then $z + \bar{z} = 0$

Important Properties of Conjugate

- (a) $\overline{(\bar{z})} = z$
- (b) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- (c) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$
- (d) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$
- (e) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}; z_2 \neq 0$
- (f) If $f(\alpha + i\beta) = x + iy \Rightarrow f(\alpha - i\beta) = x - iy$

Important Properties of Modulus

- (a) $|z| \geq 0$
- (b) $|z| \geq \text{Re}(z)$
- (c) $|z| \geq \text{Im}(z)$
- (d) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
- (e) $z\bar{z} = |z|^2$
- (f) $|z_1 z_2| = |z_1| \cdot |z_2|$
- (g) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0$
- (h) $|z^n| = |z|^n$
- (i) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2 \text{Re}(z_1 \bar{z}_2)$
 or $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \cos(\theta_1 - \theta_2)$

(j) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$

(k) $\|z_1| - |z_2| \| \leq |z_1 + z_2| \leq |z_1| + |z_2|$
 [Triangle Inequality]

(l) $\|z_1| - |z_2| \| \leq |z_1 - z_2| \leq |z_1| + |z_2|$
 [Triangle Inequality]

(m) If $\left|z + \frac{1}{z}\right| = a \ (a > 0)$, then $\max |z| = \frac{a + \sqrt{a^2 + 4}}{2}$

and $\min |z| = \frac{1}{2}(\sqrt{a^2 + 4} - a)$.

Important Properties of Amplitude

- (a) $\text{amp}(z_1 z_2) = \text{amp } z_1 + \text{amp } z_2 + 2k\pi; k \in I$
- (b) $\text{amp}\left(\frac{z_1}{z_2}\right) = \text{amp } z_1 - \text{amp } z_2 + 2k\pi; k \in I$
- (c) $\text{amp}(z^n) = n \text{amp}(z) + 2k\pi$, where proper value of k must be chosen so that RHS lies in $(-\pi, \pi]$.
- (d) $\log(z) = \log(\text{re}^{i\theta}) = \log r + i\theta = \log |z| + i \text{amp}(z)$.

Demoivre's Theorem

Case I: If n is any integer then

- (i) $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$
- (ii) $(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)(\cos \theta_3 + i \sin \theta_3)(\cos \theta_4 + i \sin \theta_4) \dots (\cos \theta_n + i \sin \theta_n) = \cos(\theta_1 + \theta_2 + \theta_3 + \dots + \theta_n) + i \sin(\theta_1 + \theta_2 + \theta_3 + \dots + \theta_n)$

Case II: If $p, q \in Z$ and $q \neq 0$ then $(\cos \theta + i \sin \theta)^{p/q}$

$$= \cos\left(\frac{2k\pi + p\theta}{q}\right) + i \sin\left(\frac{2k\pi + p\theta}{q}\right)$$

where $k = 0, 1, 2, 3 \dots q - 1$.

Cube Root of Unity

(i) The cube roots of unity are $1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$.

(ii) If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^t + \omega^{2t} = 0$; where $t \in I$ but is not the multiple of 3.

$$(c) \begin{aligned} a^2 + b^2 + c^2 - ab - bc - ca &= (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega) \\ a^3 + b^3 &= (a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega) \\ a^3 - b^3 &= (a - b)(a - \omega b)(a - \omega^2 b) \\ x^2 + x + 1 &= (x - \omega)(x - \omega^2) \end{aligned}$$

Square root of Complex Number

$$\sqrt{a + ib} = \pm \left\{ \frac{\sqrt{|z| + a}}{2} + i \frac{\sqrt{|z| - a}}{2} \right\} \text{ for } b > 0$$

$$\text{and } \pm \left\{ \frac{\sqrt{|z| + a}}{2} - i \frac{\sqrt{|z| - a}}{2} \right\} \text{ for } b < 0 \text{ where } |z| = \sqrt{a^2 + b^2}.$$

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