CHAPTER

4

Determinants

Definition

1. The determinant consisting two rows and two columns is
$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, its value is given by:

$D = a_1 b_2 - a_2 b_2$

2. A determinant which consists of three rows and three columns is called a third-order-determinant.

Let D = $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then its value is

$D = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{12}a_{21}a_{33}$

Minors and Cofactors

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then the minor M_{ij} of the element a_{ij} is the

determinant obtained by deleting the ith row and j^{th} column,

i.e. M_{11} = minor of a_{11} = $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

The **cofactor** of the element a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

Properties of Minors and Cofactors

1. The sum of the products of elements of any row (or column) of a determinant with the cofactors of the corresponding elements of any other row (or column) is zero,

i.e., if $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

$a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$ and so on.

2. The sum of the product of elements of any row (or column) of a determinant with the cofactors of the corresponding elements of the same row (or column) is Δ,

i.e., if $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

$|A| = \Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}.$

3. In general, if $|A| = \Delta$, then $|A| = \sum_{i=1}^{n} a_{ij} \, C_{ij}$ and

$|(adj\ A)| = \Delta^{n-1}$, where A is a matrix of order $n \times n$.

Properties of Determinants

1. The value of the determinant remains unchanged, if rows are changed into columns and columns are changed into rows.
e.g., $|A'| = |A|$

2. If $A = [a_{ij}]_{n \times n}$, $n > 1$ and B be the matrix obtained from A by interchanging two of its rows or columns, then
$\det(B) = -\det(A)$

3. If two rows (or columns) of a square matrix A are proportional, then $|A| = 0$.

4. $|B| = k|A|$, where B is the matrix obtained from A, by multiplying one row (or column) of A by k.

5. $|kA| = k^n |A|$, where A is a matrix of order $n \times n$.

6. If each element of a row (or column) of a determinant is the sum of two or more terms, then the determinant can be expressed as the sum of two or more determinant, e.g.,

$\begin{vmatrix} a_1 + a_2 & b & c \\ p_1 + p_2 & q & r \\ u_1 + u_2 & v & w \end{vmatrix} = \begin{vmatrix} a_1 & b & c \\ p_1 & q & r \\ u_1 & v & w \end{vmatrix} + \begin{vmatrix} a_2 & b & c \\ p_2 & q & r \\ u_2 & v & w \end{vmatrix}$

7. If the same multiple of the elements of any row (or column) of a determinant are added to the corresponding elements of any other row (or column), then the value of the new determinant remains unchanged, e.g.,
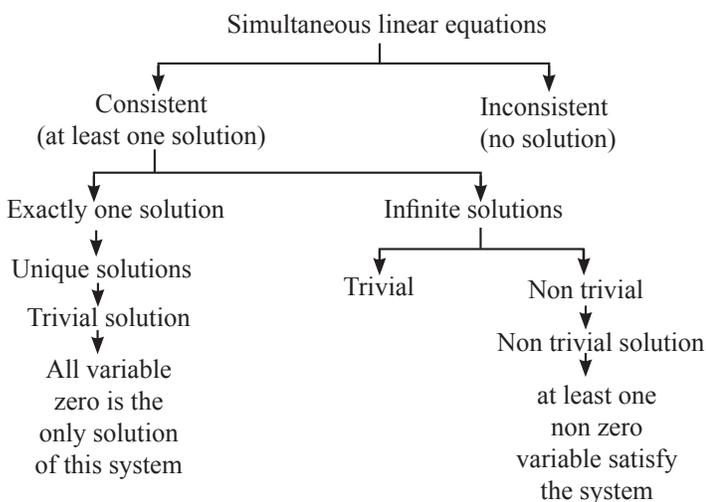
$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} + ka_{31} & a_{12} + ka_{32} & a_{13} + ka_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

8. If each element of a row (or column) of a determinant is zero, then its value is zero.

9. If any two rows (or columns) of a determinant are identical, then its value is zero.

10. If r rows (or r columns) become identical, when a is substituted for x, then $(x - a)^{r-1}$ is a factor of given determinant.

Important Results on Determinants

1. $|AB| = |A||B|$, where A and B are square matrices of the same order.

2. $|A^n| = |A|^n$.

3. If A, B and C are square matrices of the same order such that i^{th} columns (or rows) of A is the sum of ith columns (or rows) of B and C and all other columns (or rows) of A, B and C are identical, then $|A| = |B| + |C|$.

4. $|I_n| = 1$, where I_n is identity matrix of order n.

5. $|O_n| = 0$, where O_n is a zero matrix of order n.

6. If $\Delta(x)$ has a third order determinant having polynomials as its elements.

   (a) If $\Delta(a)$ has two rows (or columns) proportional, then $(x - a)$ is a factor of $\Delta(x)$.

   (b) If $\Delta(a)$ has three rows (or columns) proportional, then $(x - a)^2$ is a factor of $\Delta(x)$.

7. A square matrix A is non-singular, if $|A| \neq 0$ and singular, if $|A| = 0$.

8. Determinant of a skew-symmetric matrix of odd order is zero and of even order is a non-zero perfect square.

9. In general, $|B + C| \neq |B| + |C|$.

10. Determinant of a diagonal matrix = Product of its diagonal elements.

11. If A is a non-singular matrix, then $|A^{-1}| = \dfrac{1}{|A|} = |A|^{-1}$.

12. Determinant of a orthogonal matrix $= \pm 1$.

13. Determinant of a hermitian matrix is purely real.

14. If A and B are non-zero matrices and $AB = O$, then it implies $|A| = 0$ or $|B| = 0$.

System of Equation

Simultaneous linear equations

Consistent (at least one solution)   Inconsistent (no solution)

Exactly one solution   Infinite solutions

Unique solutions

Trivial solution

All variable zero is the only solution of this system

Trivial   Non trivial

Non trivial solution

at least one non zero variable satisfy the system

Cramer's Rule: [Simultaneous Equations Involving Three Unknowns]

Let $a_1x + b_1y + c_1z = d_1$ ...(i)

$a_2x + b_2y + c_2z = d_2$ ...(ii)

$a_3x + b_3y + c_3z = d_3$ ...(iii)

$$x = \frac{D_1}{D}, y = \frac{D_2}{D}, Z = \frac{D_3}{D}$$

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}; D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}; D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$\& D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Note:

1. If $D \neq 0$ and atleast one of D_1, D_2, $D_3 \neq 0$, then the given system of equations are consistent and have unique non trivial solution.

2. If $D \neq 0$ & $D_1 = D_2 = D_3 = 0$, then the given system of equations are consistent and have trivial solution only

3. If $D = D_1 = D_2 = D_3 = 0$, then the given system of equations are consistent and have infinite solution.

Applications of Determinants in Geometry

Let the three points in a plane be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then

1. Area of $\Delta ABC = \left| \dfrac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$

$= \left| \dfrac{1}{2} [x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \right|$

2. If the given points are collinear, then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$.

3. Let two points are $A(x_1, y_1)$, $B(x_2, y_2)$ and $P(x, y)$ be a point on the line joining points A and B, then the equation of line is given by $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$.