

CHAPTER

2

Inverse Trigonometric Functions

Principal Values and Domains of Inverse Trigonometric/circular Functions

Function	Domain	Range
(i) $y = \sin^{-1} x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
(ii) $y = \cos^{-1} x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
(iii) $y = \tan^{-1} x$	$x \in R$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
(iv) $y = \operatorname{cosec}^{-1} x$	$x \leq -1$ or $x \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}; y \neq 0$
(v) $y = \sec^{-1} x$	$x \leq -1$ or $x \geq 1$	$0 \leq y \leq \pi; y \neq \frac{\pi}{2}$
(vi) $y = \cot^{-1} x$	$x \in R$	$0 < y < \pi$

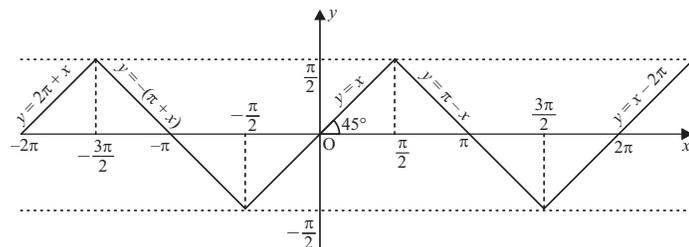
Properties of Inverse circular Functions

P-1:

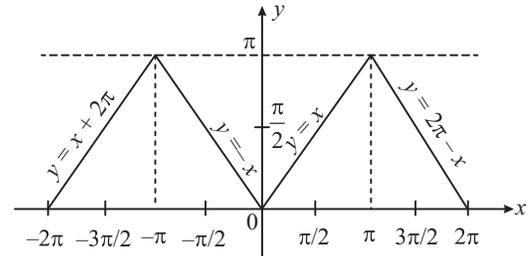
- (i) $y = \sin(\sin^{-1} x) = x, x \in [-1, 1], y \in [-1, 1], y$ is aperiodic.
- (ii) $y = \cos(\cos^{-1} x) = x, x \in [-1, 1], y \in [-1, 1], y$ is aperiodic.
- (iii) $y = \tan(\tan^{-1} x) = x, x \in R, y \in R, y$ is aperiodic.
- (iv) $y = \cot(\cot^{-1} x) = x, x \in R, y \in R, y$ is aperiodic.
- (v) $y = \operatorname{cosec}(\operatorname{cosec}^{-1} x) = x, |x| \geq 1, |y| \geq 1, y$ is aperiodic.
- (vi) $y = \sec(\sec^{-1} x) = x, |x| \geq 1, |y| \geq 1, y$ is aperiodic.

P-2:

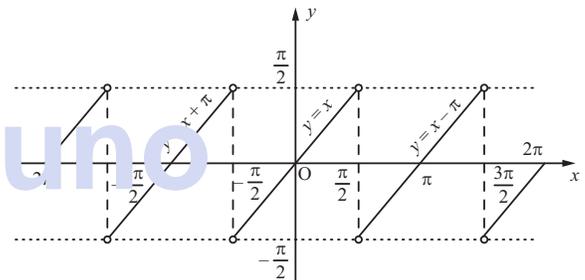
- (i) $y = \sin^{-1}(\sin x), x \in R, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Periodic with period 2π .



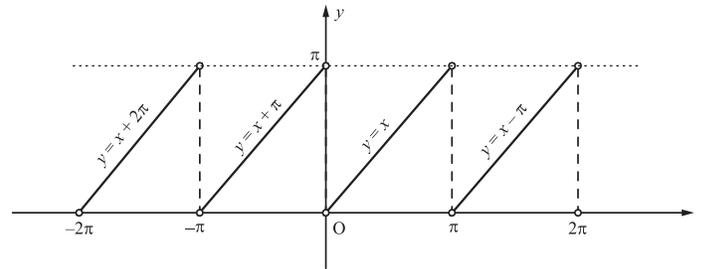
- (ii) $y = \cos^{-1}(\cos x), x \in R, y \in [0, \pi]$, periodic with period 2π .



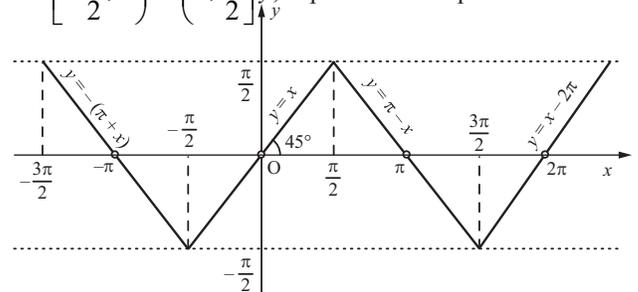
- (iii) $y = \tan^{-1}(\tan x), x \in R - \left\{\frac{n\pi}{2}\right\}, n \in I$



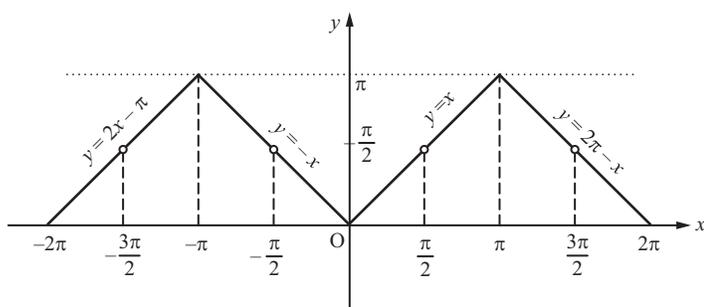
- (iv) $y = \cot^{-1}(\cot x), x \in R - \{n\pi\}, n \in I, y \in (0, \pi)$, periodic with period π .



- (v) $y = \operatorname{cosec}^{-1}(\operatorname{cosec} x), x \in R - \{n\pi\}, n \in I, y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, y is periodic with period 2π .



(vi) $y = \sec^{-1}(\sec x)$, y is periodic with period 2π
 $x \in \mathbb{R} - \left\{ (2n-1)\frac{\pi}{2} \right\}, n \in \mathbb{I}, y \in \left[0, \frac{\pi}{2} \right) \cup \left(\frac{\pi}{2}, \pi \right]$



P-3:

(i) $\operatorname{cosec}^{-1} x = \sin^{-1} \frac{1}{x}; \quad x \leq -1, x \geq 1$

(ii) $\sec^{-1} x = \cos^{-1} \frac{1}{x}; \quad x \leq -1, x \geq 1$

(iii) $\cot^{-1} x = \begin{cases} \tan^{-1} \frac{1}{x}; & x > 0 \\ \pi + \tan^{-1} \frac{1}{x}; & x < 0 \end{cases}$

P-4:

- (i) $\sin^{-1}(-x) = -\sin^{-1} x, -1 \leq x \leq 1$
- (ii) $\tan^{-1}(-x) = -\tan^{-1} x, x \in \mathbb{R}$
- (iii) $\cos^{-1}(-x) = \pi - \cos^{-1} x, -1 \leq x \leq 1$
- (iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x, x \in \mathbb{R}$
- (v) $\sec^{-1}(-x) = \pi - \sec^{-1} x, x \leq -1$ or $x \geq 1$
- (vi) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x, x \leq -1$ or $x \geq 1$

P-5:

- (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}; \quad -1 \leq x \leq 1$
- (ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}; \quad x \in \mathbb{R}$
- (iii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}; \quad |x| \geq 1$

P-6:

(i) $\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1} \frac{x+y}{1-xy}, & \text{where } x > 0, y > 0 \text{ and } xy < 1 \\ \pi + \tan^{-1} \frac{x+y}{1-xy}, & \text{where } x > 0, y > 0 \text{ and } xy > 1 \\ \frac{\pi}{2}, & \text{where } x > 0, y > 0 \text{ and } xy = 1 \end{cases}$

(ii) $\tan^{-1} x - \tan^{-1} y = \begin{cases} \tan^{-1} \frac{x-y}{1+xy}, & xy > -1 \\ \pi + \tan^{-1} \left(\frac{x-y}{1+xy} \right), & \text{where } x > 0, y > 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1} \left(\frac{x-y}{1+xy} \right), & \text{where } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$

(iii) $\sin^{-1} x + \sin^{-1} y = \sin^{-1} (x\sqrt{1-y^2} + y\sqrt{1-x^2}),$
 where $x \geq 0, y \geq 0$ & $(x^2 + y^2) < 1$

Note that: $x^2 + y^2 < 1 \Rightarrow 0 < \sin^{-1} x + \sin^{-1} y < \frac{\pi}{2}$

(iv) $\sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1} (x\sqrt{1-y^2} + y\sqrt{1-x^2}),$
 where $x > 0, y > 0$ and $x^2 + y^2 > 1$.

Note that: $x^2 + y^2 > 1 \Rightarrow \frac{\pi}{2} < \sin^{-1} x + \sin^{-1} y < \pi$.

(v) $\sin^{-1} x - \sin^{-1} y = \sin^{-1} [x\sqrt{1-y^2} - y\sqrt{1-x^2}]$
 where $x > 0, y > 0$.

(vi) $\cos^{-1} x + \cos^{-1} y = \cos^{-1} (xy - \sqrt{1-x^2}\sqrt{1-y^2}); \quad x, y \geq 0$

(vii) $\cos^{-1} x - \cos^{-1} y = \begin{cases} \cos^{-1} (y\sqrt{1-x^2} - x\sqrt{1-y^2}), & x > 0, y > 0 \text{ and } x < y \\ -\cos^{-1} (xy + \sqrt{1-x^2}\sqrt{1-y^2}), & x > 0, y > 0 \text{ and } x > y \end{cases}$

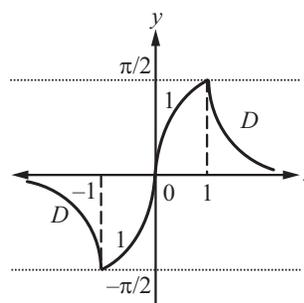
(viii) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1} \left[\frac{x+y+z-xyz}{1-xy-yz-zx} \right]$

if $x > 0, y > 0, z > 0$ & $xy + yz + zx < 1$.

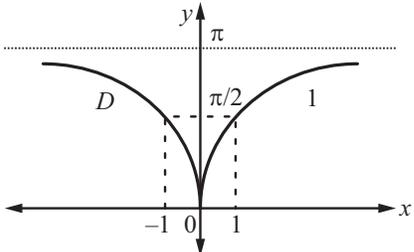
Note that: In the above results x & y are taken positive. In case if these are given as negative, we first apply P-4 and then use above results.

Simplified Inverse Trigonometric Functions

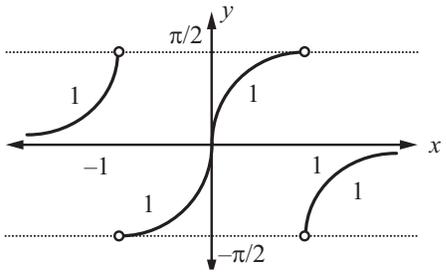
(a) $y = f(x) = \sin^{-1} \left(\frac{2x}{1+x^2} \right) = \begin{cases} 2 \tan^{-1} x & \text{if } |x| \leq 1 \\ \pi - 2 \tan^{-1} x & \text{if } x > 1 \\ -(\pi + 2 \tan^{-1} x) & \text{if } x < -1 \end{cases}$



$$(b) y = f(x) = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right) = \begin{cases} 2 \tan^{-1} x & \text{if } x \geq 0 \\ -2 \tan^{-1} x & \text{if } x < 0 \end{cases}$$

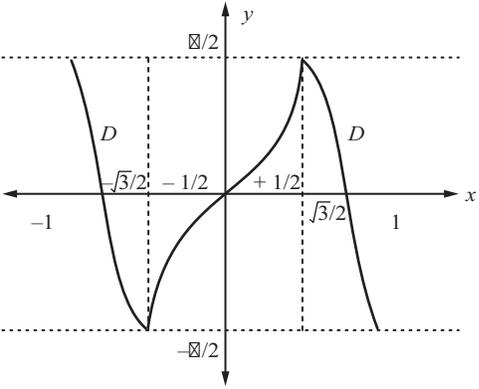


$$(c) y = f(x) = \tan^{-1} \frac{2x}{1-x^2} = \begin{cases} 2 \tan^{-1} x & \text{if } |x| < 1 \\ \pi + 2 \tan^{-1} x & \text{if } x < -1 \\ -(\pi - 2 \tan^{-1} x) & \text{if } x > 1 \end{cases}$$



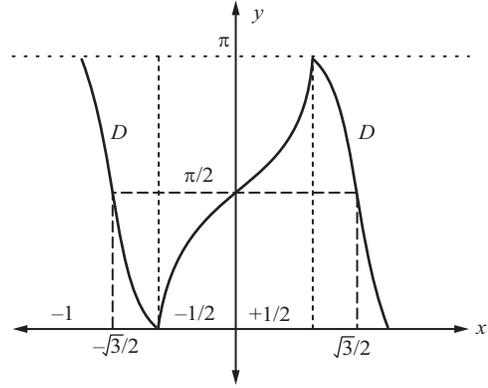
$$(d) y = f(x) = \sin^{-1} (3x - 4x^3)$$

$$= \begin{cases} -(\pi + 3 \sin^{-1} x) & \text{if } -1 \leq x \leq -\frac{1}{2} \\ 3 \sin^{-1} x & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ \pi - 3 \sin^{-1} x & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

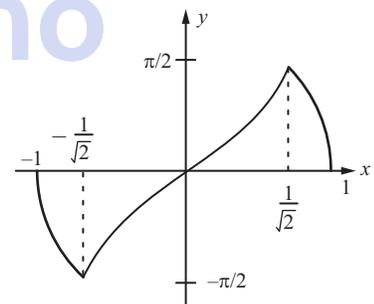


$$(e) y = f(x) = \cos^{-1} (4x^3 - 3x)$$

$$= \begin{cases} 3 \cos^{-1} x - 2\pi & \text{if } -1 \leq x \leq -\frac{1}{2} \\ 2\pi - 3 \cos^{-1} x & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 3 \cos^{-1} x & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$



$$(f) \sin^{-1} (2x\sqrt{1-x^2}) = \begin{cases} -(\pi + 2 \sin^{-1} x) & -1 \leq x \leq -\frac{1}{\sqrt{2}} \\ 2 \sin^{-1} x & -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}} \\ \pi - 2 \sin^{-1} x & \frac{1}{\sqrt{2}} \leq x \leq 1 \end{cases}$$



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