CHAPTER 5

Limits of Functions

Limit

Limit of a function $f(x)$ is said to exist as $x \to a$ when,

$$\underset{\substack{x\to a^-\\ \text{(Left hand limit)}}}{\lim} f(a-h) = \underset{\substack{x\to a^+\\ \text{(Right hand limit)}}}{\lim} f(a+h) = M \text{ some finite value } M.$$

Indeterminate Forms

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad (\infty)-(\infty)$$

$$\infty \times 0, \quad (1)^{\infty}, \quad (0)^0, \quad (\infty)^0$$

Standard Limits

$$\lim_{x\to0}\frac{\sin x}{x} = \lim_{x\to0}\frac{\tan x}{x} = \lim_{x\to0}\frac{\tan^{-1}x}{x} = \lim_{x\to0}\frac{\sin^{-1}x}{x}$$

$$= \lim_{x\to0}\frac{e^x-1}{x} = \lim_{x\to0}\frac{ln(1+x)}{x} = 1,$$

$$\lim_{x\to0}(1+x)^{1/x} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e, \lim_{x\to0}\frac{a^x-1}{x} = \log_e a, a>0,$$

$$\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}.$$

Note

$$\boxed{\underset{a>1}{\log_a x} <<< \underset{a>1}{a^x} <<< \underset{x\in N}{x!}}$$

Fundamental Theorems on Limits

Let $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$. If l and m exists finitely then:

(*a*) Sum rule: $\lim_{x\to a}\left[f(x)+g(x)\right] = l+m$

(*b*) Difference rule: $\lim_{x\to a}\left[f(x)-g(x)\right] = l-m$

(*c*) Product rule: $\lim_{x\to a}\left[f(x)\cdot g(x)\right] = l.m$

(*d*) Quotient rule: $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{l}{m}$, provided $m \neq 0$

(*e*) Power rule: If m and n are integers, then

$\lim_{x\to a}\left[f(x)\right]^{m/n} = l^{m/n}$, provided $l^{m/n}$ is a real number.

(*f*) $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(m)$; provided $f(x)$ is continuous at $x = m$.

Limits Using Expansion

(*i*) $a^x = 1+\frac{xlna}{1!}+\frac{x^2ln^2a}{2!}+\frac{x^3ln^3a}{3!}+\ldots, a>0$

(*ii*) $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$, for $-1<x\le1$

(*iii*) $ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$, for $-1<x\le1$

(*iv*) $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots, x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(*v*) $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots, x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(*vi*) $\tan x = x+\frac{x^3}{3}+\frac{2x^5}{15}+\ldots, x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(*vii*) $\sin^{-1}x = x+\frac{1^2}{3!}x^3+\frac{1^2\cdot3^2}{5!}x^5+\frac{1^2\cdot3^2\cdot5^2}{7!}x^7+\ldots$

(*viii*) $\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$

(*xi*) For $|x|<1, n\in R, (1+x)^n = 1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+\ldots\infty$

(*xii*) $(1+x)^{1/x} = e^{\frac{1}{x}ln(1+x)} = e\left[1-\frac{x}{2}+\frac{11}{24}x^2-\frac{21}{48}x^3+\ldots+\infty\right]$

Limits of form 1^∞, 0^0, ∞^0.

Also for $(1)^\infty$ type of problems we can use following rules.

(a) $\lim\limits_{x\to 0}(1+x)^{1/x} = e$,

(b) $\lim\limits_{x\to a}[f(x)]^{g(x)}$, where $f(x)\to 1; g(x)\to\infty$ as $x\to a$ then

$$\lim_{x\to a}[f(x)]^{g(x)} = e^{\lim\limits_{x\to a}\{f(x)-1\}g(x)}$$

Sandwich Theorem or Squeeze Play Theorem

If $f(x) \le g(x) \le h(x)\ \forall\ x$ and $\lim\limits_{x\to a} f(x) = l = \lim\limits_{x\to a} h(x)$, then $\lim\limits_{x\to a} g(x) = l$