

CHAPTER

21



Mathematical Reasoning

Mathematical Statements

A statement is a sentence which is either true or false, but not both simultaneously.

Example

1. "I'm happy today !" is not a statement
2. "New delhi is in India", it is a statement
3. "x is a natural number", is not a statement

Simple Statement

A statement is simple if it can not be broken down into two or more statement.

Compound Statement

A compound statement is a statement which is made up of two or more statements. The individual statements are called component statements.

E.g., "All rational numbers are real and all real numbers are complex". Is a compound statement.

The two component statements are:

- p : All rational numbers are real.
 q : All real numbers are complex.

Negation of a Statement

The denial of a statement is called the negation of the statement.

If p is a statement, then the negation of p is also a statement and is denoted by $\sim p$ and read as 'not p '.

E.g., Consider a statement:

p : sky is blue

Its negation is:

$\sim p$: sky is not blue"

Connectives

The word 'and' (conjunction) (\wedge)

Any two component statements can be connected by the word 'and' to form a compound statement.

The rules for the compound statement with 'and' are:

Rule 1: The compound statement with 'and' is true if all its component statements are true.

Rule 2: The compound statement with 'and' is false if some or all of its component statements are false.

The word 'or' (disjunction) (\vee)

Any two component statements can be connected by the word "or" to form a compound statement.

The rules for the compound statement with 'or' are:

Rule 1: A compound statement with an 'or' is true if one component statement is true or all the component statements are true.

Rule 2: A compound statement with an 'or' is false if all the component statements are false.

Quantifiers

In mathematics, sometimes we come across many mathematical statements containing phrases "There exists" and "For every". These two phrases are called **quantifiers**. Phrase "for every (or for all)" is called the **universal quantifier** and the phrase "There exists" is known as the **existential quantifier**.

E.g., \forall : For every real number x , x is less than $x + 1$.

\exists : There exists a continent for every country in the world.

Implications

The statements of the form "If p then q ", " p only if q ", and "if and only if" are called implications.

p implies q is denoted by $p \Rightarrow q \equiv \sim p \vee q$

if and only if is denoted by the symbol \Leftrightarrow

General Rules for Validating Statements

Rule 1: If p and q are mathematical statements, then in order to show that the statement " p and q " is true, the following steps are followed.

Case I: Show that the statement p is true.

Case II: Show that the statement q is true.

Rule 2: In order to show that the statement " p or q " is true, one must consider the following.

Case I: By assuming that p is false, show that q must be true.

Case II: By assuming that q is false, show that p must be true.

Rule 3: Statements with 'If-then'

In order to prove the statement "If p then q " we need to show that any of the following cases is true.

Case I: By assuming that p is true, show that q must be true.

Case II: By assuming that q is false, show that p must be false.

Rule 4: Statements with 'If and only if'

In order to prove the statement “ p if and only if q ”, we need to show.

- (i) If p is true, then q is true and
- (ii) If q is true, then p is true.

Method of Contradiction

To check whether a statement p is true, we assume that p is not true *i.e.* $\sim p$ is true. Then, we arrive at some result which is contradictory to our assumption. Therefore, we conclude that p is true.

This method involves giving an example of a situation where the statement is not valid. Such an example is called a counter example.

Important /Critical Points to Remember

Truth Table

A table that shows the relationship between the truth value of compound statement. $S(p, q, r, \dots)$ and the truth values of its sub-statements p, q, r, \dots , etc., is called the truth table of statement S . If p and q are two simple statements then truth table for basic logical connectives of:

Conjunction		
p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction		
p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Negation	
p	$\sim p$
T	F
F	T

Conditional		
p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$ or $(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Logical Equivalence

Two compound statements $S_1(p_1, q_1, r_1, \dots)$ and $S_2(p_2, q_2, r_2, \dots)$ are said to be logically equivalent if they have the same truth values for all logical possibilities that is identical truth table.

If statements S_1 and S_2 are equivalent then we write $S_1 \equiv S_2$

Tautology: A statement is said to be a tautology if it is true for all logical possibilities.

Contradiction: A statement is a contradiction if it is false for all logical possibilities *i.e.* its truth value is always F .

Duality

Two compound statements S_1 and S_2 are said to be duals of each other if one can be obtained from the other by replacing \wedge by \vee and \vee by \wedge .

Converse: The converse of the conditional statement $p \rightarrow q$ is defined as $q \rightarrow p$.

Inverse: The inverse of the conditional statement $p \rightarrow q$ is defined as $\sim p \rightarrow \sim q$

Contrapositive: The contrapositive of the conditional statement $p \rightarrow q$ is defined as $\sim q \rightarrow \sim p$

Negation of Compound Statements

- (a) **Negation of conjunction:** If p and q are two statements then $\sim(p \wedge q) \equiv \sim p \vee \sim q$
- (b) **Negation of disjunction:** If p and q are two statements then $\sim(p \vee q) \equiv \sim p \wedge \sim q$
- (c) **Negation of implication:** If p and q are two statements, then $\sim(p \Rightarrow q) \equiv p \wedge \sim q$
- (d) **Negation of Biconditional:** If p and q are two statements, then $\sim(p \Leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

Algebra of Statements

If p, q, r are any three statements then some law of algebra of statements are as follows:

- (a) **Idempotent Laws**
 - (i) $p \vee p \equiv p$
 - (ii) $p \wedge p \equiv p$
- (b) **Commutative Laws**
 - (i) $p \vee q \equiv q \vee p$
 - (ii) $p \wedge q \equiv q \wedge p$
- (c) **Associative Law**
 - (i) $(p \vee q) \vee r \equiv p \vee (q \vee r)$
 - (ii) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- (d) **Distributive Laws**
 - (i) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
 - (ii) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- (e) **De‘Morgan’s Law**
 - (i) $\sim(p \wedge q) \equiv \sim p \vee \sim q$
 - (ii) $\sim(p \vee q) \equiv \sim p \wedge \sim q$
- (f) **Contrapositive Laws**

For any statement p , we have

$$p \Rightarrow q \equiv \sim q \Rightarrow \sim p$$
- (g) **Involution Laws (Double Negation Laws)**

$$\sim(\sim p) \equiv p$$