CHAPTER 7

Quadratic Equations

Solution of Quadratic Equation & Relation Between Roots & Co-Efficients

(*a*) The solutions of the quadratic equation, $ax^2 + bx + c = 0$ is given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(*b*) The expression $b^2 - 4ac = D$ is called the discriminant of the quadratic equation.

(*c*) If α & β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then;

- (*i*) $\alpha + \beta = -b/a$
- (*ii*) $\alpha\beta = c/a$
- (*iii*) $|\alpha - \beta| = \sqrt{D}/|a|$

(*d*) Quadratic equation whose roots are α & β is $(x - \alpha)(x - \beta) = 0$ *i.e.*, $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ *i.e.* x^2 – (sum of roots)x + product of roots = 0.

Nature of Roots

(*a*) Consider the quadratic equation $ax^2+bx+c=0$ where $a, b, c \in R$ & $a \neq 0$ then;

- (*i*) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal).
- (*ii*) $D = 0 \Leftrightarrow$ roots are real & coincident (equal).
- (*iii*) $D < 0 \Leftrightarrow$ roots are imaginary.
- (*iv*) If $p + iq$ is one root of a quadratic equation, then the other root must be the conjugate $p - iq$ & vice versa. ($p, q \in R$ & $i = \sqrt{-1}$).

(*b*) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in Q$ & $a \neq 0$ then;

- (*i*) If D is a perfect square, then roots are rational.
- (*ii*) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then other root will be $p - \sqrt{q}$. (if a, b, c are rational). Because the coefficients are real

Common Roots of Two Quadratic Equations

(*a*) Only one common root.

Let α be the common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$ then $a\alpha^2 + b\alpha + c = 0$ & $a'\alpha^2 + b'\alpha + c' = 0$. By Cramer's Rule

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - ac'} = \frac{1}{ab' - a'b}$$

Therefore, $\alpha = \dfrac{ca' - c'a}{ab' - a'b} = \dfrac{bc' - b'c}{a'c - ac'}$

So the condition for a common root is

$(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$

(*b*) If both roots are same then $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$

Roots Under Particular Cases

Let the quadratic equation $ax^2 + bx + c = 0$ has real roots and

(*a*) If $b = 0 \Rightarrow$ roots are of equal magnitude but of opposite sign

(*b*) If $c = 0 \Rightarrow$ one roots is zero other is $-b/a$

(*c*) If $a = c \Rightarrow$ roots are reciprocal to each other

(*d*) If $\left.\begin{array}{l} a > 0\ c < 0 \\ a < 0\ c > 0 \end{array}\right\} \Rightarrow$ roots are of opposite signs.

(*e*) If $\left.\begin{array}{l} a > 0, b > 0, c > 0 \\ a < 0, b < 0, c < 0 \end{array}\right\} \Rightarrow$ both roots are negative.

(*f*) If $\left.\begin{array}{l} a > 0, b < 0, c > 0 \\ a < 0, b > 0, c < 0 \end{array}\right\} \Rightarrow$ both roots are positive.

(*g*) If sign of a = sign of $b \neq$ sign of $c \Rightarrow$ Greater root in magnitude is negative.

(*h*) If sign of b = sign of $c \neq$ sign of $a \Rightarrow$ Greater root in magnitude is positive.

(*i*) If $a + b + c = 0 \Rightarrow$ one root is 1 and second root is c/a.

Maximum & Minimum Values of Quadratic Expression

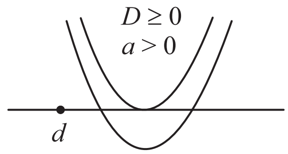
Maximum & Minimum Values of expression $y = ax^2 + bx + c$ is $\dfrac{-D}{4a}$ which occurs at $x = -(b/2a)$ according as $a < 0$ or $a > 0$.

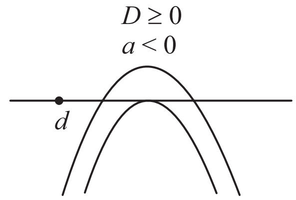
$y \in \left[\dfrac{-D}{4a}, \infty\right)$ if a > 0 & $y \in \left(-\infty, \dfrac{-D}{4a}\right]$ if a < 0.

Location of Roots

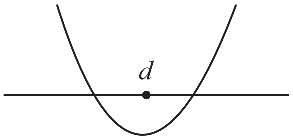
Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$, $a \neq 0$

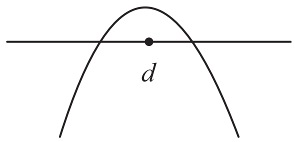
(*a*) Conditions for both the roots of f(x) = 0 to be greater than a specified number '*d*' are $\boldsymbol{D \geq 0;\ a.f(d) > 0\ \&\ (-b/2a) > d}$.



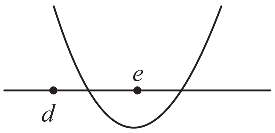


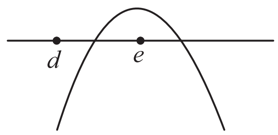
(*b*) Conditions for the both roots of $f(x) = 0$ to lie on either side of the number 'd' in other words the number 'd' lies between the roots of $f(x) = 0$ is $\boldsymbol{a.f(d) < 0}$.





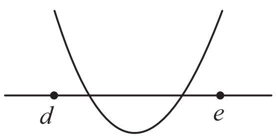
(*c*) Conditions for exactly one root of $f(x) = 0$ to lie in the interval (d,e) *i.e.*, $d < x < e$ is $\boldsymbol{f(d).\, f(e) < 0}$.

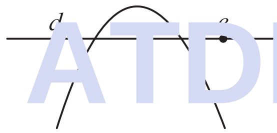




(*d*) Conditions that both roots of $f(x) = 0$ to be confined between the numbers d & e are (here $d < e$).

$D \geq 0$; $a.f(d) > 0$ & $af(e) > 0$; $d < (-b/2a) < e$





General Quadratic Expression in two Variables

$f(x, y) = ax^2 + 2\,hxy + by^2 + 2gx + 2\,fy + c$ may be resolved into two linear factors if;

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \text{ OR } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Theory of Equations

If α_1, α_2, α_3, ...α_n are the roots of the equation; $f(x) = a_0\, x^n + a_1\, x^{n-1} + a_2\, x^{n-2} + \ldots\ldots + a_{n-1}\, x + a_n = 0$ where $a_0, a_1, \ldots\ldots a_n$ are constants $a_0 \neq 0$ then,

$$\sum \alpha_1 = -\frac{a_1}{a_0}, \sum \alpha_1\alpha_2 = +\frac{a_2}{a_0}, \sum \alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$$

Note:

(*i*) Every odd degree equation has at least one real root whose sign is opposite to that of its last term, when coefficient of highest degree term is (+)ve {If not then make it (+)ve}.

Ex. $x^3 - x^2 + x - 1 = 0$

(*ii*) Even degree polynomial whose last term is (–)ve & coefficient of highest degree term is (+)ve has atleast two real roots, one (+)ve & one (–)ve.

(*iii*) If equation contains only even power of x & all coefficient are (+)ve, then all roots are imaginary.