

# CHAPTER

# 1



# Relations and Functions

## Relations

If  $A$  and  $B$  are two non-empty sets, then a relation  $R$  from  $A$  to  $B$  is a subset of  $A \times B$ .

### Representation of a Relation

**Roster form:** In this form, we represent the relation by the set of all ordered pairs belongs to  $R$ .

**Set-builder form:** In this form, we represent the relation  $R$  from set  $A$  to set  $B$  as

$R = \{(a, b) : a \in A, b \in B \text{ and the rule which relate the elements of } A \text{ and } B\}$ .

### Domain, Codomain and Range of a Relation

Let  $R$  be a relation from a non-empty set  $A$  to a non-empty set  $B$ . Then, set of all first components or coordinates of the ordered pairs belonging to  $R$  is called the domain of  $R$ , while the set of all second components or coordinates of the ordered pairs belonging to  $R$  is called the range of  $R$ . Also, the set  $B$  is called the codomain of relation  $R$ .

Thus, domain of  $R = \{a : (a, b) \in R\}$  and range of  $R = \{b : (a, b) \in R\}$

### Types of Relations

**Empty or Void Relation:** As  $\phi \subset A \times A$ , for any set  $A$ , so  $\phi$  is a relation on  $A$ , called the empty or void relation.

**Universal Relation:** Since,  $A \times A \subseteq A \times A$ , so  $A \times A$  is a relation on  $A$ , called the universal relation.

**Identity Relation:** The relation  $I_A = \{(a, a) : a \in A\}$  is called the identity relation on  $A$ .

**Reflexive Relation:** A relation  $R$  on a set  $A$  is said to be reflexive relation, if every element of  $A$  is related to itself.

Thus,  $(a, a) \in R, \forall a \in A \Rightarrow R$  is reflexive.

**Symmetric Relation:** A relation  $R$  on a set  $A$  is said to be symmetric relation iff  $(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$

i.e.  $a R b \Rightarrow b R a, \forall a, b \in A$

**Transitive Relation:** A relation  $R$  on a set  $A$  is said to be transitive relation, iff  $(a, b) \in R$  and  $(b, c) \in R$

$\Rightarrow (a, c) \in R, \forall a, b, c \in A$

## Equivalence Relation

A relation  $R$  on a set  $A$  is said to be an equivalence relation, if it is simultaneously reflexive, symmetric and transitive on  $A$ .

## Functions

Let  $A$  and  $B$  be two non-empty sets, then a function  $f$  from set  $A$  to set  $B$  is a rule which associates each element of  $A$  to a unique element of  $B$ .

### Domain, Codomain and Range of a Function

If  $f : A \rightarrow B$  is a function from  $A$  to  $B$ , then

- (i) the set  $A$  is called the domain of  $f(x)$ .
- (ii) the set  $B$  is called the codomain of  $f(x)$ .
- (iii) the subset of  $B$  containing only the images of elements of  $A$  is called the range of  $f(x)$ .

### Number of Functions

Let  $X$  and  $Y$  be two finite sets having  $m$  and  $n$  elements respectively. Then each element of set  $X$  can be associated to any one of  $n$  elements of set  $Y$ . So, total number of functions from set  $X$  to set  $Y$  is  $n^m$ .

### Number of One-One Functions

Let  $A$  and  $B$  are finite sets having  $m$  and  $n$  elements respectively, then the number of one-one functions from  $A$  to  $B$  is

$$= \begin{cases} n(n-1)(n-2)\dots(n-(m-1)), & n \geq m \\ 0, & n < m \end{cases}$$

### Number of Onto (or Surjective) Functions

Let  $A$  and  $B$  are finite sets having  $m$  and  $n$  elements respectively, then number of onto (or surjective) functions from  $A$  to  $B$  is

$$= \begin{cases} n^m - {}^n C_1(n-1)^m + {}^n C_2(n-2)^m - {}^n C_3(n-3)^m + \dots, & n < m \\ n!, & n = m \\ 0, & n > m \end{cases}$$

### Number of Bijective Functions

Let  $A$  and  $B$  are finite sets having  $m$  and  $n$  elements respectively, then number of bijective functions from  $A$  to  $B$  is

$$= \begin{cases} n!, & \text{if } n = m \\ 0, & \text{if } n > m \text{ or } n < m \end{cases}$$

## Properties of Greatest Integer Function

- (i)  $[x + n] = n + [x], n \in I$
- (ii)  $[-x] = -[x], x \in I$
- (iii)  $[-x] = -[x] - 1, x \notin I$
- (iv)  $[x] \geq n \Rightarrow x \geq n, n \in I$
- (v)  $[x] > n \Rightarrow x \geq n + 1, n \in I$
- (vi)  $[x] \leq n \Rightarrow x < n + 1, n \in I$
- (vii)  $[x] < n \Rightarrow x < n, n \in I$
- (viii)  $[x + y] = [x] + [y + x - [x]]$  for all  $x, y \in R$
- (ix)  $[x + y] \geq [x] + [y]$

## Important Points To Be Remembered

- (i) Constant function is periodic with no fundamental period.
- (ii) If  $f(x)$  is periodic with period  $T$ , then  $\frac{1}{f(x)}$  and  $\sqrt{f(x)}$  are also periodic with same period  $T$ .
- (iii) If  $f(x)$  is periodic with period  $T$ , then  $kf(ax + b)$  is periodic with period  $\frac{T}{|a|}$ , where  $a, b, k \in R$  and  $a, k \neq 0$ .

## Properties of Even and Odd Functions

- (i)  $gof$  or  $fog$  is even, if both  $f$  and  $g$  are even or if  $f$  is odd and  $g$  is even or if  $f$  is even and  $g$  is odd.
- (ii)  $gof$  or  $fog$  is odd, if both of  $f$  and  $g$  are odd.

- (iii) If  $f(x)$  is an even function, then  $\frac{d}{dx}f(x)$  is an odd function and if  $f(x)$  is an odd function, then  $\frac{d}{dx}f(x)$  is an even function.
- (iv) The graph of an even function is symmetrical about  $Y$ -axis.
- (v) The graph of an odd function is symmetrical about origin or symmetrical in opposite quadrants.
- (vi) An even function can never be one-one, however an odd function may or may not be one-one.

## Properties of Inverse Function

- (a) The inverse of a bijection is unique.
- (b) If  $f: A \rightarrow B$  is a bijection and  $g: B \rightarrow A$  is the inverse of  $f$ , then  $fog = I_B$  and  $gof = I_A$ , where  $I_A$  &  $I_B$  are identity functions on the sets  $A$  &  $B$  respectively. If  $fog = I$ , then  $f$  is inverse of itself.
- (c) The inverse of a bijection is also a bijection.
- (d) If  $f$  &  $g$  are two bijections  $f: A \rightarrow B, g: B \rightarrow C$  &  $gof$  exist, then the inverse of  $gof$  also exists and  $(gof)^{-1} = f^{-1}og^{-1}$ .
- (e) The graph of  $f^{-1}$  obtained by reflecting the graph of  $f$  about the line  $y = x$ .

## General

If  $x, y$  are independent variables, then :

- (a)  $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$
- (b)  $f(xy) = f(x) \cdot f(y) \Rightarrow f(x) = x^n, n \in R$  or  $f(x) = 0$
- (c)  $f(x + y) = f(x) \cdot f(y) \Rightarrow f(x) = a^{kx}$  or  $f(x) = 0$
- (d)  $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$ , where  $k$  is a constant.