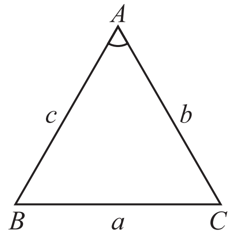
CHAPTER

25 Solutions of Triangles

1. Sine Rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$



2. Cosine Formula:

(*i*) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

(*ii*) $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$

(*iii*) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

3. Projection Formula

(*i*) $a = b\cos C + c\cos B$

(*ii*) $b = c\cos A + a\cos C$

(*iii*) $c = a\cos B + b\cos A$

4. Napier's Analogy - Tangent Rule

(*i*) $\tan\dfrac{B-C}{2} = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$

(*ii*) $\tan\dfrac{C-A}{2} = \dfrac{c-a}{c+a}\cot\dfrac{B}{2}$

(*iii*) $\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

5. Trigonometric Functions of Half Angles

(*i*) $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$; $\sin\dfrac{B}{2} = \sqrt{\dfrac{(s-c)(s-a)}{ca}}$;

$\sin\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$

(*ii*) $\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$; $\cos\dfrac{B}{2} = \sqrt{\dfrac{s(s-b)}{ca}}$; $\cos\dfrac{C}{2} = \sqrt{\dfrac{s(s-c)}{ab}}$

(*iii*) $\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} = \dfrac{\Delta}{s(s-a)}$ where $s = \dfrac{a+b+c}{2}$ is semi perimetre of triangle.

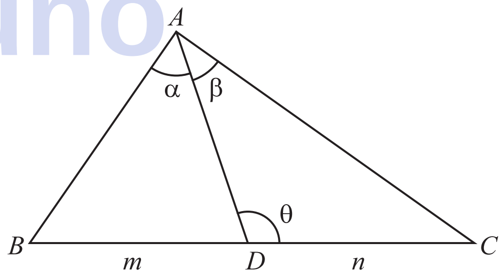
(*iv*) $\sin A = \dfrac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \dfrac{2\Delta}{bc}$

6. Area of Triangle (Δ)

$$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$$

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

7. m-n Rule



If $BD : DC = m : n$, then

$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

$= n\cot B - m\cot C$

8. Radius of Circumcircle

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$$

9. Radius of The Incircle

(*i*) $r = \dfrac{\Delta}{s}$

(*ii*) $r = (s-a)\tan\dfrac{A}{2} = (s-b)\tan\dfrac{B}{2} = (s-c)\tan\dfrac{C}{2}$

(*iii*) $r = \dfrac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$ and so on

(iv) $r = 4R \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

10. Radius of The Ex-Circles

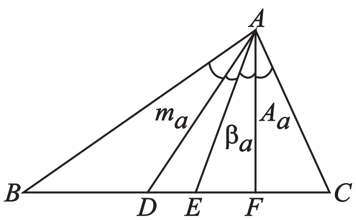
(i) $r_1 = \frac{\Delta}{s-a}; r_2 = \frac{\Delta}{s-b}; r_3 = \frac{\Delta}{s-c}$

(ii) $r_1 = s\tan\frac{A}{2}; r_2 = s\tan\frac{B}{2}; r_3 = s\tan\frac{C}{2}$

(iii) $r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$ and so on.

(iv) $r_1 = 4R\sin\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$

11. Length of Angle Bisectors, Medians and Altitudes



(i) Length of an angle bisector from the angle $A = \beta_a = \frac{2bc\cos\frac{A}{2}}{b+c}$.

(ii) Length of median from angle $A = m_a = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$.

(iii) Length of altitude from the angle $A = A_a = \frac{2\Delta}{a}$.

12. The Distances of the special Points from Vertices and Sides of Triangle

(i) Circumcentre (O) : $OA = R$ and $O_a = R\cos A$

(ii) Incentre (I) : $IA = r\,\text{cosec}\,\frac{A}{2}$ and $I_a = r$

(iii) Excentre (I_1) : $I_1A = r_1\,\text{cosec}\,\frac{A}{2}$

(iv) Orthocentre : $HA = 2R\cos A$ & $H_a = 2R\cos B\cos C$

(v) Centroid (G) : $GA = \frac{1}{3}\sqrt{2b^2+2c^2-a^2}$ and $G_a = \frac{2\Delta}{3a}$

13. Orthocentre and Pedal Triangle

The triangle KLM which is formed by joining the feet of the altitudes is called the Pedal Triangle.

(i) Its angles are $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$.

(ii) Its sides are $a\cos A = R\sin 2A$,

$b\cos B = R\sin 2B$ and

$c\cos C = R\sin 2C$

(iii) Circumradii of the triangles PBC, PCA, PAB and ABC are equal.

Where P is orthocenter of ΔABC.

14. Excentral Triangle

The triangle formed by joining the three excentres I_1, I_2 and I_3 of ΔABC is called the excentral or excentric triangle.

(i) ΔABC is the pedal triangle of the $\Delta I_1I_2I_3$.

(ii) Its angles are $\frac{\pi}{2}-\frac{A}{2}, \frac{\pi}{2}-\frac{B}{2}$ and $\frac{\pi}{2}-\frac{C}{2}$.

(iii) Its sides are $4R\cos\frac{A}{2}$, $4R\cos\frac{B}{2}$ and $4R\cos\frac{C}{2}$.

(iv) $I I_1 = 4R\sin\frac{A}{2}; I I_2 = 4R\sin\frac{B}{2}; I I_3 = 4R\sin\frac{C}{2}$.

(v) Incentre I or ΔABC is the orthocentre of the excentral $\Delta I_1I_2I_3$.

15. Distance Between Special Points

(i) Distance between circumcentre and orthocentre

$OH^2 = R^2(1 - 8\cos A\cos B\cos C)$

(ii) Distance between circumcentre and incentre

$OI^2 = R^2\left(1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = R^2 - 2Rr$

(iii) Distance between circumcentre and centroid

$OG^2 = R^2 - \frac{1}{9}(a^2+b^2+c^2)$