CHAPTER 5

Trigonometric Equations

1. **Solution of Trigonometric Equation:** A solution of a trigonometric equation is the value of an unknown angle that satisfies the equation.

   Thus, the trigonometric equation may has infinite number of solutions and are classified as:

   Principal Solution: The solution of the trigonometric equation lying in the interval $[0, 2\pi)$.

   General Solution: Since all the trigonometric functions are many one & periodic, hence there are infinite values for which trigonometric functions have the same value. All such possible values are given by a general formula. Such general formula is called a general solution of the trigonometric equation.

2. **General Solutions of Some Trigonometric Equations (To be Remembered)**

   (*a*) If $\sin\theta = 0$ then $\theta = n\pi$, $n \in I$ (set of integers)

   (*b*) If $\cos\theta = 0$ then $\theta = (2n + 1)\pi/2$, $n \in I$

   (*c*) If $\tan\theta = 0$ then $\theta = n\pi$, $n \in I$

   (*d*) If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right], n \in I$

   (*e*) If $\cos\theta = \cos\alpha$ then $\theta = 2n\pi \pm \alpha$, $n \in I$, $\alpha \in [0, \pi]$

   (*f*) If $\tan\theta = \tan\alpha$ then $\theta = n\pi + \alpha$, $n \in I$, $\alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

   (*g*) If $\sin\theta = 1$ then $\theta = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}, n \in I$

   (*h*) If $\cos\theta = 1$ then $\theta = 2n\pi$, $n \in I$

   (*i*) If $\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$ or $\tan^2\theta = \tan^2\alpha$ then $\theta = n\pi \pm \alpha$, $n \in I$

   (*j*) For $n \in I$, $\sin n\pi = 0$

   $\sin(n\pi + \theta) = (-1)^n \sin\theta$

   (*k*) $\cos n\pi = (-1)^n$, $n \in I$

   $\cos(n\pi + \theta) = (-1)^n \cos\theta$

3. **Solution of Different Types of Trigonometric Equations**

   (*i*) General solution of equation $a\cos\theta + b\sin\theta = c$

   Consider $a\sin\theta + b\cos\theta = c$ ...(*i*)

   $$\therefore \frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta = \frac{c}{\sqrt{a^2+b^2}}$$

   Equation (*i*) has the solution only if $|c| \le \sqrt{a^2+b^2}$

   Let $\frac{a}{\sqrt{a^2+b^2}} = \cos\phi, \frac{b}{\sqrt{a^2+b^2}} = \sin\phi, \phi = \tan^{-1}\frac{b}{a}$

   by introducing this auxiliary argument ϕ, equation (*i*) reduces to $\sin(\theta+\phi) = \frac{c}{\sqrt{a^2+b^2}}$

   Now this equation can be solved easily.

   (*ii*) **General Solution of Equation of Form**

   $a_0\sin^n x + a_1\sin^{n-1}x\cos x + a_2\sin^{n-2}x\cos^2 x + ... + a_n\cos^n x = 0$

   $a_0, a_1, ..., a_n$ are real numbers.

   Such an equation is solved by dividing equation by $\cos^n x$

   (*iii*) **Solving Equations using Boundedness:**

   (*a*) Trigonometric equations involving a single variable

   Step 1: When LHS and RHS of equation have their ranges say R_1 and R_2 in common domain D and $R_1 \cap R_2 = \phi$, then the equation has no solution.

   Step 2: $R_1 \cap R_2$ have finitely many elements and the number of elements are very few, then the individual cases can be analyzed and solved.

   (*b*) Trigonometric equations involving more than one variable. To solve an equation involving more than one variable, definite solutions can be obtained if the extreme values (range) of the functions are used.

   (*iv*) **Solving equation by changing the variable or by substitution method:**

   (*a*) Equation of the form $P(\sin x \pm \cos x, \sin x . \cos x) = 0$, when $P(y, z)$ is a polynomial, can be solved by the substitution: $\cos x \pm \sin x = t$.

   (*b*) Equation of the form of $a\sin x + b\cos x + d = 0$, where a, b & d are the real numbers that can be solved by changing $\sin x$ & $\cos x$ into their corresponding tangent of the half-angle.

   (*c*) Many equations can be solved by introducing a new variable.

   (*v*) **Solution of Trigonometric Equation using Graphs**

   Solution of $f(x) - g(x) = 0$ can be find out by application of following steps:

   Step 1: Write the equation $f(x) = g(x)$.

   Step 2: Draw the graph of $y = f(x)$ and $y = g(x)$ on same x-y plane.

Step 3: The number of points of intersection of $f(x)$ and $g(x)$ are same as the number of solutions of $f(x) - g(x) = 0$

(*vi*) **Solving a System of Trigonometric Equations**

To solve the simultaneous equations in one variable, say x, we observe the following steps

Step 1: Find the values of x satisfying both the equations individually and lying in $[0, 2\pi)$

Step 2: Select the values satisfying both the equations simultaneously.

Step 3: Generalize the values to get the general solution.

(*vii*) **Trigonometric Inequalities:** To solve the trigonometric inequalities of the type $f(x) \leq a$ or $f(x) \geq a$ where $f(x)$ is some trigonometric equation, we take the following steps

Step 1: Draw the graph of $f(x)$ in an interval length equal to the fundamental period of $f(x)$.

Step 2: Draw the line $y = a$.

Step 3: Take the portion of the graph for which the inequality is satisfied.

Step 4: To generalize, add $p \times n$, $n \in I$ in the final solution where p is the fundamental period of $f(x)$.