CHAPTER 4

Trigonometric Ratios and Identities

1. Trigonometric Ratio – Basic Terminology

hypotenuse, opposite, θ, adjacent C
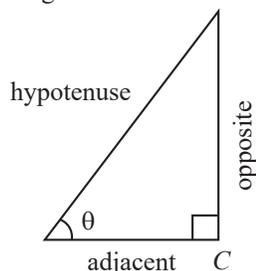
The six trigonometric ratios of θ are defined as follows:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}}, \quad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}, \quad \sec\theta = \frac{\text{hypotenus}}{\text{adjacent}}$$

2. Trigonometric identities:
   (i) $\sin^2\theta + \cos^2\theta = 1$
   (ii) $\sec^2\theta - \tan^2\theta = 1$
   (iii) $\csc^2\theta - \cot^2\theta = 1$

3. Allied angles:

   Two angles are said to be allied when their sum or difference is either zero or a multiple of $\frac{\pi}{2}$, two angles x, y are allied angles iff $|x \pm y| = 0 \ or \ \frac{n\pi}{2}, n \in N$.

q→	$\frac{\pi}{2}-\theta$	$\frac{\pi}{2}+\theta$	$\pi-\theta$	$\pi+\theta$	$\frac{3\pi}{2}-\theta$	$\frac{3\pi}{2}+\theta$	$2\pi-\theta$	$2\pi+\theta$	$-\theta$
sin	cos θ	cos θ	sin θ	–sin θ	–cos θ	–cos θ	–sin θ	sin θ	–sin θ
cos	sin θ	–sin θ	–cos θ	–cos θ	–sin θ	sin θ	cos θ	cos θ	cos θ
tan	cot θ	–cot θ	– tan θ	tan θ	cot θ	–cot θ	–tan θ	tan θ	–tan θ
cot	tan θ	–tan θ	– cot θ	cot θ	tan θ	–tan θ	–cot θ	cot θ	–cot θ
sec	cosecθ	–cosecθ	–sec θ	–sec θ	–cosec θ	cosecθ	sec θ	sec θ	sec θ
cosec	sec θ	sec θ	cosec θ	–cosec θ	–sec θ	–sec θ	–cosec θ	cosec θ	–cosec θ

4. Sum & Difference Formula
   (i) $\sin(A+B) = \sin A \cos B + \cos A \sin B$
   (ii) $\sin(A-B) = \sin A \cos B - \cos A \sin B$
   (iii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$
   (iv) $\cos(A-B) = \cos A \cos B + \sin A \sin B$
   (v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
   (vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$
   (vii) $\cot(A+B) = \dfrac{\cot B \cot A - 1}{\cot B + \cot A}$
   (viii) $\cot(A-B) = \dfrac{\cot B \cot A + 1}{\cot B - \cot A}$

5. Product to sum
   (i) $2\sin A \cos B = \sin(A+B) + \sin(A-B)$
   (ii) $2\cos A \sin B = \sin(A+B) - \sin(A-B)$
   (iii) $2\cos A \cos B = \cos(A+B) + \cos(A-B)$
   (iv) $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

6. Trigonometric transformations:
   (i) $\sin C + \sin D = 2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$
   (ii) $\sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$
   (iii) $\cos C + \cos D = 2\cos\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

(iv) $\cos C - \cos D = -2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$

$\qquad\qquad = 2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{D-C}{2}\right)$

Some useful Formulae:

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(iii) $\tan(A+B)\tan(A-B) = \dfrac{\tan^2 A - \tan^2 B}{1-\tan^2 A \tan^2 B}$

(iv) $\cot(A+B)\cot(A-B) = \dfrac{\cot^2 B \cot^2 A - 1}{\cot^2 B - \cot^2 A}$

7. Some standard trigonometric values

$\dfrac{Angle \rightarrow}{Trigonometric\ Function \downarrow}$	15°	18°	$22\frac{1}{2}°$	36°
sin	$\dfrac{\sqrt{3}-1}{2\sqrt{2}}$	$\dfrac{\sqrt{5}-1}{4}$	$\dfrac{1}{2}\sqrt{2-\sqrt{2}}$	$\dfrac{\sqrt{10-2\sqrt{5}}}{4}$
cos	$\dfrac{\sqrt{3}+1}{2\sqrt{2}}$	$\dfrac{\sqrt{10+2\sqrt{5}}}{4}$	$\dfrac{1}{2}\sqrt{2+\sqrt{2}}$	$\dfrac{\sqrt{5}+1}{4}$
tan	$2-\sqrt{3}$	$\dfrac{\sqrt{25-10\sqrt{5}}}{5}$	$\sqrt{2}-1$	$\sqrt{5-2\sqrt{5}}$

8. Double Angle / Triple Angle

(i) $\sin 2A = 2\sin A \cos A$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

Suppose that A is not an odd multiple of $\dfrac{\pi}{2}$. Then

(iii) $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$

(iv) $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

(v) $\tan 2 = \dfrac{2\tan}{1-\tan^2}$ (Here $2A$ is also not an odd multiple of $\dfrac{\pi}{2}$)

(vi) $\sin 3A = 3\sin A - 4\sin^3 A, \forall A \in R$

(vii) $\cos 3A = 4\cos^3 A - 3\cos A, \forall A \in R$

(viii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$ (3A, A are not odd multiples of $\dfrac{\pi}{2}$)

(ix) $\cot 3A \ \dfrac{3\cot A \ \cot^3 A}{1 \ 3\cot^2 A}$ (3A, A are not multiples of $\dfrac{\pi}{2}$)

9. Half Angle

(i) $\sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2}$

(ii) $\cos A = \cos^2\dfrac{A}{2} - \sin^2\dfrac{A}{2} = 2\cos^2\dfrac{A}{2} - 1 = 1 - 2\sin^2\dfrac{A}{2}$

(iii) $\tan A = \dfrac{2\tan\dfrac{A}{2}}{1-\tan^2\dfrac{A}{2}}$ $\left(\text{where } \dfrac{A}{2} \neq (2n+1)\dfrac{\pi}{2}, n \in Z\right)$

10. Conditional Identities

In a Triangle If $A + B + C = \pi$, then

(i) $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

(ii) $\sin A + \sin B + \sin C = 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

(iii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$

(iv) $\sin^2 A + \sin^2 B + \sin^2 C = 2(1 + \cos A \cos B \cos C)$

(v) $\cos^2 A + \cos^2 B + \cos^2 C = (1 - 2\cos A \cos B \cos C)$

11. Domains, Ranges and Periodicity of Trigonometric Functions:

T-Ratio	Domain	Range	Period
sin x	R	$[-1, 1]$	2π
cos x	R	$[-1, 1]$	2π
tan x	$R - \left\{(2n+1)\dfrac{\pi}{2}\right\}; n \in I$	R	π
cot x	$R - \{n\pi: n \in I\}$	R	π
sec x	$R - \left\{(2n+1)\dfrac{\pi}{2}\right\}; n \in I$	$(-\infty, -1] \cup [1, \infty)$	2π
cosec x	$R - \{n\pi: n \in I\}$	$(-\infty, -1] \cup [1, \infty)$	2π

12. Maxima-Minima

(i) $a\cos\theta + b\sin\theta$ will always lie in the interval $\left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$ i.e., the maximum and minimum values are $\sqrt{a^2+b^2}, -\sqrt{a^2+b^2}$ respectively.

(ii) Minimum value of $a^2\tan^2\theta + b^2\cot^2\theta$ is $2ab$ where $a, b > 0$.

(iii) $-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} < a\cos(\alpha+\theta) + b\cos(\beta+\theta) \leq \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$ where α and β are known angles.

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(*iv*) If $\alpha, \beta, \in \left(0, \dfrac{\pi}{2}\right)$ and $\alpha + \beta = \sigma$ (constant) then

    (*a*) Maximum value of the expression $\cos \alpha \cos \beta$, $\cos \alpha + \cos \beta$, $\sin \alpha \sin \beta$ or $\sin \alpha + \sin \beta$ occurs when $\alpha = \beta = \dfrac{\sigma}{2}$

    (*b*) Minimum value of $\sec \alpha + \sec \beta$, $\tan \alpha + \tan \beta$, $\operatorname{cosec} \alpha + \operatorname{cosec} \beta$ occurs when $\alpha = \beta = \dfrac{\sigma}{2}$

(*v*) If *A, B, C* are the angles of a triangle then maximum value of

$\sin A + \sin B + \sin C$ and $\sin A \sin B \sin C$ occurs when $A = B = C = 60°$

(*vi*) In case a quadratic in $\sin \theta$ & $\cos \theta$ is given then the maximum or minimum values can be obtained by making perfect square.

13. **Sum of Three or More Angles**

  (*i*) $\sin (A + B + C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C = \sum \sin A \cos B \cos C - \prod \sin A$

  (*ii*) $\cos (A + B + C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \sin C = \prod \cos A - \sum \cos A \sin B \sin C$

  (*iii*) $\tan (A + B + C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

$$= \frac{\sum \tan A - \prod \tan A}{1 - \sum \tan A \tan B}$$

14. **Summation of Series**

$\sin \alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + ... + \sin(\alpha + \overline{n-1}\beta)$

$$= \frac{\sin\left(\alpha + \dfrac{n-1}{2}\beta\right) \sin \dfrac{n\beta}{2}}{\sin \dfrac{\beta}{2}}$$

$\cos \alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + ... + \cos(\alpha + \overline{n-1}\beta)$

$$= \frac{\cos\left(\alpha + \dfrac{n-1}{2}\beta\right) \sin \dfrac{n\beta}{2}}{\sin \dfrac{\beta}{2}}$$