

CHAPTER

12

Vector Algebra

Important Definitions

- ❖ **Representation of Vectors:** A vector \vec{a} is represented by the directed line segment \overline{AB} . The magnitude of the vector \vec{a} is equal to $|\overline{AB}|$, and the direction of the vector \vec{a} is along the line from A to B .
- ❖ **Scalar Quantity:** A quantity that has only magnitude and is not related to any direction is called a scalar quantity.
- ❖ **Vector Quantity:** A quantity that has magnitude and also a direction in space is called a vector quantity.
- ❖ **Null Vector or Zero Vector:** If the initial and terminal points of a vector coincide, then it is called a zero vector. It is denoted by $\vec{0}$ or O . Its magnitude is zero and direction indeterminate.
- ❖ **Unit Vector:** A vector whose magnitude is of unit length along my vector \vec{a} is called a unit vector in the direction of \vec{a} and is denoted by \hat{a}
- ❖ **Equal Vector:** Two non-zero vectors are said to be equal vectors if their magnitude is equal and directions are the same.
- ❖ **Collinear Vector:** Two or more non-zero vectors are said to be collinear vectors if they are parallel to the same line.
- ❖ **Like and Unlike Vector:** Collinear vectors having the same direction are known as like vectors, while those having opposite directions are known as, unlike vectors.
- ❖ **Coplanar Vector:** Two or more non-zero vectors are said to be coplanar vectors if these are parallel to the same plane.
- ❖ **Localised Vector and Free Vector:** A vector drawn parallel to a given vector through a specified point as the initial point, is known as a localised vector. If the initial point of a vector is not specified, it is said to be a free vector.
- ❖ **Position Vector:** Let O be the origin and A be a point such that $\overline{OA} = \vec{a}$, then we say that the position vector of A is \vec{a} .

Negative of a Vector

- ❖ Let \overline{AB} be a vector directed from A to B . then $-\overline{AB}$ is a vector which would be directed from B to A .

Coinitial Vectors

- ❖ Two vectors are said to be coinitial vectors if both the vectors have the same initial points.

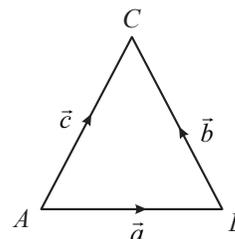
Co-terminal Vectors

- ❖ Two vectors are said to be Co-terminal vectors if both the vectors have the same terminating point.

Algebra of vectors

Addition of Vectors

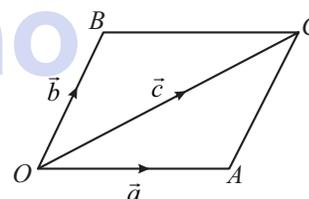
Triangle Law



$$\text{Result: } \vec{a} + \vec{b} = \vec{c} \text{ or } \overline{AB} + \overline{BC} = \overline{AC}$$

Converse of triangle law is also true.

Parallelogram Law



$$\text{Result: } \vec{a} + \vec{b} = \vec{c} \text{ or } \overline{OA} + \overline{OB} = \overline{OC}$$

Properties of vector addition:

- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (commutative)
- $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (associative)
- $\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}$
- $\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$
- $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$
- $|\vec{a} - \vec{b}| \geq ||\vec{a}| - |\vec{b}||$

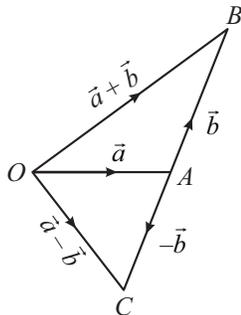
Multiplication of Vector by Scalars

If \vec{a} and \vec{b} are vectors & m, n are scalars, then

- $m(\vec{a}) = (\vec{a})m = m\vec{a}$
- $m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$
- $(m+n)\vec{a} = m\vec{a} + n\vec{a}$
- $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$

Subtraction of Vectors

In the given diagram \vec{a} and \vec{b} are represented by \vec{OA} and \vec{AB} . We extend the line AB in opposite direction upto C , where $AB = AC$. The line segment \vec{AC} will represent the vector $-\vec{b}$. By joining the points O and C , the vector represented by \vec{OC} is $\vec{a} + (-\vec{b})$. i.e., denotes the vector $\vec{a} - \vec{b}$.



Note:

- (i) $\vec{a} - \vec{a} = \vec{a} + (-\vec{a}) = \vec{0}$
- (ii) $\vec{a} - \vec{b} \neq \vec{b} - \vec{a}$
Hence subtraction of vectors does not obey the commutative law.
- (iii) $\vec{a} - (\vec{b} - \vec{c}) \neq (\vec{a} - \vec{b}) - \vec{c}$
i.e. subtraction of vectors does not obey the associative law.

Important Properties and Formulae

- ❖ If $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ then $\vec{r}_1 + \vec{r}_2 = (x_1 + x_2)\hat{i} + (y_1 + y_2)\hat{j} + (z_1 + z_2)\hat{k}$ and $\vec{r}_1 = r_2 \Leftrightarrow x_1 = x_2, y_1 = y_2, z_1 = z_2$.
- ❖ \vec{a} and \vec{b} are parallel or collinear if $\vec{a} = m\vec{b}$ and only if for some non-zero scalar m .
- ❖ $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$ or $\vec{a} = |\vec{a}|\hat{a}$
- ❖ $\vec{r}, \vec{a}, \vec{b}$ are coplanar if and only if $\vec{r} = x\vec{a} + y\vec{b}$ for some scalars x and y .
- ❖ If the position vectors of the points A and B be \vec{a} and \vec{b} then, the position vectors of the points dividing the line AB in the ratio $m : n$ internally and externally are $\frac{m\vec{b} + n\vec{a}}{m + n}$ and $\frac{m\vec{b} - n\vec{a}}{m - n}$, respectively.
- ❖ If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$
- ❖ Given vectors $x_1\vec{a} + y_1\vec{b} + z_1\vec{c}, x_2\vec{a} + y_2\vec{b} + z_2\vec{c}, x_3\vec{a} + y_3\vec{b} + z_3\vec{c}$, where $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, will be coplanar if and only if $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$

- ❖ $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$
- ❖ $|\vec{a} - \vec{b}| \geq ||\vec{a}| - |\vec{b}||$

Scalar Product or Dot Product

- ❖ $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos \theta$, where $0 \leq \theta \leq \pi$
- ❖ If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
- ❖ If \vec{a} and \vec{b} are the non-zero vectors, then $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$
- ❖ $\cos \theta = \frac{|\vec{a} \cdot \vec{b}|}{|\vec{a}||\vec{b}|}$ where θ is the acute angle made by \vec{a} with \vec{b}
- ❖ Projection of \vec{b} along $\vec{a} = \frac{\vec{b} \cdot \vec{a}}{|\vec{a}|}$
- ❖ Component of a vector \vec{r} in the direction of \vec{a} and perpendicular to \vec{a} are $\left(\frac{\vec{r} \cdot \vec{a}}{|\vec{a}|^2}\right)\vec{a}$ and $\vec{r} - \left\{\frac{\vec{r} \cdot \vec{a}}{|\vec{a}|^2}\right\}\vec{a}$ respectively.
- ❖ $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ and $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{i} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{j} = \hat{i} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

Vector Product

- ❖ The product of vectors \vec{a} and \vec{b} and is denoted by $\vec{a} \times \vec{b} = (|\vec{a}||\vec{b}|\sin \theta)\hat{n}$
- ❖ $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- ❖ If $\vec{a} \parallel \vec{b}$ or \vec{c} is parallel to \vec{b} , then $\sin \theta = 0$ and so $\vec{a} \times \vec{b} = \vec{0}$
- ❖ Distributive laws: $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ and $(\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a}$
- ❖ If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then
 - (i) $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$
 - (ii) $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
- ❖ If two vectors \vec{a} and \vec{b} are parallel, then $\theta = 0$ or π i.e. $\sin \theta = 0$ in both cases.
- ❖ Two vectors \vec{a} and \vec{b} are parallel if their corresponding components are proportional.
- ❖ Area of the triangle $ABC = \frac{1}{2} |\vec{AB} \times \vec{AC}|$
 $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}, \hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$
- ❖ Unit vector perpendicular to the plane of \vec{a} and \vec{b} is $\hat{n} = \pm \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$
- ❖ If θ is the angle between \vec{a} and \vec{b} , then $\sin \theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$

Scalar Triple Product

❖ If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and

$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}, \text{ then } (\vec{a} \times \vec{b}) \cdot \vec{c} = [\vec{a} \ \vec{b} \ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- ❖ $[\vec{a} \ \vec{b} \ \vec{c}] = [\vec{b} \ \vec{c} \ \vec{a}] = [\vec{c} \ \vec{b} \ \vec{a}]$ but $[\vec{a} \ \vec{b} \ \vec{c}] = -[\vec{a} \ \vec{c} \ \vec{b}]$ etc.
- ❖ If any two of the vectors $\vec{a}, \vec{b}, \vec{c}$ are equal, then $[\vec{a} \ \vec{b} \ \vec{c}] = 0$.
- ❖ The position of dot and cross in a scalar triple product can be interchanged. Hence, $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$
- ❖ The value of a scalar triple product is zero if two of its vectors are parallel.
- ❖ $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if $[\vec{a} \ \vec{b} \ \vec{c}] = 0$.
- ❖ Volume of the parallelepiped whose coterminous edges are formed by $\vec{a}, \vec{b}, \vec{c} = |[\vec{a} \ \vec{b} \ \vec{c}]|$.
- ❖ Volume of a tetrahedron with three coterminous edges $\vec{a}, \vec{b}, \vec{c} = \frac{1}{6} |[\vec{a} \ \vec{b} \ \vec{c}]|$.
- ❖ Volume of prism on a triangular base with three coterminous edges $\vec{a}, \vec{b}, \vec{c} = \frac{1}{2} |[\vec{a} \ \vec{b} \ \vec{c}]|$.
- ❖ In particular $\hat{i} \cdot (\hat{j} \times \hat{k}) = 1$
 $[\hat{i} \ \hat{j} \ \hat{k}] = 1$
- ❖ $[K \vec{a} \ \vec{b} \ \vec{c}] = K[\vec{a} \ \vec{b} \ \vec{c}]$
- ❖ $[(\vec{a} + \vec{b}) \ \vec{c} \ \vec{d}] = [\vec{a} \ \vec{c} \ \vec{d}] + [\vec{b} \ \vec{c} \ \vec{d}]$
- ❖ $[\vec{a} - \vec{b} \ \vec{b} - \vec{c} \ \vec{c} - \vec{a}] = 0$ and $[\vec{a} + \vec{b} \ \vec{b} + \vec{c} \ \vec{c} + \vec{a}] = 2[\vec{a} \ \vec{b} \ \vec{c}]$

$$❖ [\vec{a} \ \vec{b} \ \vec{c}]^2 = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} & \vec{a} \cdot \vec{c} \\ \vec{b} \cdot \vec{a} & \vec{b} \cdot \vec{b} & \vec{b} \cdot \vec{c} \\ \vec{c} \cdot \vec{a} & \vec{c} \cdot \vec{b} & \vec{c} \cdot \vec{c} \end{vmatrix} = [\vec{a} \times \vec{b} \ \vec{b} \times \vec{c} \ \vec{c} \times \vec{a}]$$

Vector Triple Product

- ❖ If $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then $(\vec{a} \times \vec{b}) \times \vec{c}$ and $\vec{a} \times (\vec{b} \times \vec{c})$ are known as vector triple product.
- ❖ $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ and $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$

- ❖ $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector in the plane of vectors \vec{b} and \vec{c} .
- ❖ The vector triple product is not commutative i.e., $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

$$❖ \text{Lagrange's identity: } (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

$$= (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$$

$$❖ (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a} \ \vec{b} \ \vec{d}]\vec{c} - [\vec{a} \ \vec{b} \ \vec{c}]\vec{d}$$

$$= [\vec{c} \ \vec{d} \ \vec{a}]\vec{b} - [\vec{c} \ \vec{d} \ \vec{b}]\vec{a}$$

Distance between Lines

(i) If two parallel lines are given by

$\vec{r}_1 = \vec{a}_1 + K\vec{b}$ and $\vec{r}_2 = \vec{a}_2 + K\vec{b}$, then distance (d) between them is given by

$$d = \frac{|\vec{b} \times (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}|}$$

$$\text{Shortest Distance} = \frac{|\vec{AB} \cdot (\vec{p} \times \vec{q})|}{|\vec{p} \times \vec{q}|} = \frac{|(\vec{b} - \vec{a}) \cdot (\vec{p} \times \vec{q})|}{|\vec{p} \times \vec{q}|}$$

The two lines directed along \vec{p} and \vec{q} will intersect only if shortest distance = 0.

Reciprocal System of Vectors

- ❖ If $\vec{a}, \vec{b}, \vec{c}$ be any three non-coplanar vectors so that $[\vec{a} \ \vec{b} \ \vec{c}] \neq 0$ then the three vectors $\vec{a}', \vec{b}', \vec{c}'$ defined by the equations $\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a} \ \vec{b} \ \vec{c}]}$, $\vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a} \ \vec{b} \ \vec{c}]}$, $\vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a} \ \vec{b} \ \vec{c}]}$ are called the reciprocal system of vectors to the given vectors $\vec{a}, \vec{b}, \vec{c}$.

❖ **Properties of Reciprocal system of vectors:**

(i) $\vec{a} \cdot \vec{a}' = \vec{b} \cdot \vec{b}' = \vec{c} \cdot \vec{c}' = 1$

(ii) $[\vec{a} \ \vec{b} \ \vec{c}][\vec{a}' \ \vec{b}' \ \vec{c}'] = 1$

(iii) $\vec{i}' = \vec{i}, \vec{j}' = \vec{j}, \vec{k}' = \vec{k}$

(iv) If $\{\vec{a}', \vec{b}', \vec{c}'\}$ is reciprocal system of $\{\vec{a}, \vec{b}, \vec{c}\}$ and \vec{r} is any vector, then

$$\vec{r} = (\vec{r} \cdot \vec{a}')\vec{a} + (\vec{r} \cdot \vec{b}')\vec{b} + (\vec{r} \cdot \vec{c}')\vec{c}'$$

$$\vec{r} = (\vec{r} \cdot \vec{a})\vec{a}' + (\vec{r} \cdot \vec{b})\vec{b}' + (\vec{r} \cdot \vec{c})\vec{c}$$

